ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $\omega \to$ Brocard's angle the following relationship holds :

$$\frac{1}{\sin\omega} \ge 2.\frac{m_a+m_b+m_c}{h_a+h_b+h_c}$$

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$$\text{Since } \frac{1}{\sin\omega} = \frac{\sqrt{\sum_{cyc} a^2b^2}}{2rs} \text{ and } \because \left(\sum_{cyc} m_a\right)^2 \overset{\text{Chu-Yang}}{\le} 4s^2 - 16Rr + 5r^2$$

$$\therefore \text{ in order to prove : } \frac{1}{\sin\omega} \overset{?}{\ge} 2.\frac{m_a+m_b+m_c}{h_a+h_b+h_c}, \text{ it suffices to prove :}$$

$$((s^2+4Rr+r^2)^2 - 16Rrs^2)(s^2+4Rr+r^2)^2 \overset{?}{\ge} 64R^2r^2s^2(4s^2-16Rr+5r^2)$$

$$\Leftrightarrow s^8 + 4r^2s^6 - r^2(288R^2 - 16Rr - 6r^2)s^4 +$$

$$r^3(1024R^3 - 256R^2r + 32Rr^2 + 4r^3)s^2 + r^4(4R+r)^4 \overset{?}{\underset{①}{\ge}} 0 \text{ and } \because$$

$$(s^2-16Rr+5r^2)^4 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ in order to prove ①, it suffices to prove :}$$

$$\text{LHS of ①} \overset{?}{\ge} (s^2-16Rr+5r^2)^4 \Leftrightarrow (4R-r)s^6 - r(114R^2 - 61Rr + 9r^2)s^4 +$$

$$r^2(1088R^3 - 976R^2r + 302Rr^2 - 31r^3)s^2 -$$

$$r^3(4080R^4 - 5136R^3r + 2394R^2r^2 - 501Rr^3 + 39r^4) \overset{?}{\underset{②}{\ge}} 0 \text{ and } \because$$

$$(4R-r)(s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ in order to prove ②, it suffices to prove :}$$

$$\text{LHS of ②} \overset{?}{\ge} (4R-r)(s^2-16Rr+5r^2)^3 \Leftrightarrow (78R^2 - 47Rr + 6r^2)s^4 -$$

$$r(1984R^3 - 1712R^2r + 478Rr^2 - 44r^3)s^2 +$$

$$r^2(12304R^4 - 14320R^3r + 6246R^2r^2 - 1199Rr^3 + 86r^4) \overset{?}{\underset{③}{\ge}} 0 \text{ and } \because$$

$$(78R^2 - 47Rr + 6r^2)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ in order to prove ③,}$$

$$\text{it suffices to prove : LHS of ③} \overset{?}{\ge} (78R^2 - 47Rr + 6r^2)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (128R^3 - 143R^2r + 46Rr^2 - 4r^3)s^2 \overset{?}{\underset{④}{\ge}} r\begin{pmatrix} 1916R^4 - 2548R^3r + \\ 1190R^2r^2 - 234Rr^3 + 16r^4 \end{pmatrix}$$

$$\text{Finally, LHS of ④} \overset{\text{Gerretsen}}{\ge} (128R^3 - 143R^2r + 46Rr^2 - 4r^3)(16Rr - 5r^2) \overset{?}{\ge}$$

$$\text{RHS of ④} \Leftrightarrow 132t^4 - 380t^3 + 261t^2 - 60t + 4 \overset{?}{\ge} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(132t^3 - 116t^2 + 29t - 2) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2 \Rightarrow ④ \Rightarrow ③ \Rightarrow ②$$

$$\Rightarrow \text{①} \textbf{ is true} \therefore \frac{1}{\sin\omega} \geq 2.\frac{m_a + m_b + m_c}{h_a + h_b + h_c}\ \forall\, \Delta\, ABC,$$

$'' ='' $ **iff** Δ **ABC is equilateral (QED)**