ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$max\left\{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}},\frac{a+b}{a+c}+\frac{a+c}{a+b}\right\}\le\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\le\frac{w_b}{w_c}+\frac{w_c}{w_b}$$

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$$\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{B}{2}}-2=\frac{\left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)^2}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}=\frac{\left(\frac{(s-c)(s-a)}{ca}-\frac{(s-a)(s-b)}{ab}\right)^2}{\frac{(s-c)(s-a)}{ca}\cdot\frac{(s-a)(s-b)}{ab}}$$

$$=\frac{\left(b(s-c)-c(s-b)\right)^2}{bc(s-b)(s-c)}\Rightarrow\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{B}{2}}-2=\frac{s^2(b-c)^2}{bc(s-b)(s-c)}\rightarrow(m)\ \&\ \text{now,}$$

$$\frac{a+b}{a+c}+\frac{a+c}{a+b}\overset{?}{\le}\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\overset{\text{squaring}}{\Leftrightarrow}\left(\frac{a+b}{a+c}\right)^2+\left(\frac{a+c}{a+b}\right)^2\overset{?}{\le}\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}$$

$$\overset{\text{squaring}}{\Leftrightarrow}\left(\frac{a+b}{a+c}\right)^4+\left(\frac{a+c}{a+b}\right)^4-2\overset{?}{\le}\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{B}{2}}-2\overset{\text{via }(m)}{\Leftrightarrow}$$

$$\frac{((a+b)^4-(c+a)^4)^2}{(a+b)^4(c+a)^4}\overset{?}{\le}\frac{s^2(b-c)^2}{bc(s-b)(s-c)}$$

$$\Leftrightarrow\frac{((a+b)^2+(c+a)^2)^2(2a+b+c)^2(b-c)^2}{(a+b)^4(c+a)^4}\overset{?}{\le}\frac{s^2(b-c)^2}{bc(s-b)(s-c)}\Leftrightarrow$$

$$bc(s-b)(s-c)((a+b)^2+(c+a)^2)^2(2a+b+c)^2\overset{?}{\le}s^2(a+b)^4(c+a)^4$$

$$(\because(b-c)^2\ge0)\ \textit{and}\ \because bc(s-b)(s-c)\overset{\text{AM-GM}}{\le}\frac{(b+c)^2a^2}{16}\ \therefore\ \text{it suffices to prove :}$$

$$4s(a+b)^2(c+a)^2\overset{?}{\ge}a(b+c)((a+b)^2+(c+a)^2)(2a+b+c)\Leftrightarrow$$

$$4\left(\sum_{cyc}x\right)(2z+x+y)^2(2y+z+x)^2\overset{?}{\ge}$$

$$(y+z)(2x+y+z)((2z+x+y)^2+(2y+z+x)^2)(2(y+z)+2x+y+z)$$

$$(x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y)\Leftrightarrow$$

$$4x^5+20x^4(y+z)+36x^3(y+z)^2+8x^3yz+26x^2(y+z)^3+40x^2yz(y+z)+$$
$$6x(y+z)^4+56xyz(y+z)^2+4xy^2z^2+(y+z)^5+22yz(y+z)^3+4y^2z^2(y+z)$$

$$\overset{?}{\ge}0\rightarrow\text{true (strict inequality)}\Rightarrow\frac{a+b}{a+c}+\frac{a+c}{a+b}\overset{(1)}{\le}\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}$$

$$\text{Again, } \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}} \overset{?}{\leq} \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \overset{\text{squaring twice and via (m)}}{\Leftrightarrow} \frac{s^2(b-c)^2}{bc(s-b)(s-c)} \overset{?}{\geq} \frac{(b^2-c^2)^2}{b^2c^2}$$

$$\Leftrightarrow \left(\sum_{cyc} x\right)^2 (z+x)(x+y) \overset{?}{\geq} yz(2x+y+z)^2 \; (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow x^4+3x^3(y+z)+3x^2(y^2+yz+z^2)+x(y^3+z^3+y^2z+yz^2) \overset{?}{\geq} 0$$

$$\to \text{true (strict inequality)} \therefore \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}} \overset{②}{\leq} \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}$$

$$\text{Finally, } \frac{w_b}{w_c}+\frac{w_c}{w_b} = \frac{\frac{2a.4R\cos\frac{C}{2}\sin\frac{C}{2}}{c+a}.\cos\frac{B}{2}}{\frac{2a.4R\cos\frac{B}{2}\sin\frac{B}{2}}{a+b}.\cos\frac{C}{2}}+\frac{\frac{2a.4R\cos\frac{B}{2}\sin\frac{B}{2}}{a+b}.\cos\frac{C}{2}}{\frac{2a.4R\cos\frac{C}{2}\sin\frac{C}{2}}{c+a}.\cos\frac{B}{2}}$$

$$= \frac{\left(\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right)^2}{\frac{c+a}{a+b}}+\frac{\left(\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}\right)^2}{\frac{a+b}{c+a}} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right)^2}{\frac{c+a}{a+b}+\frac{a+b}{c+a}} \overset{\text{via ①}}{\geq}$$

$$\frac{\left(\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right)\left(\frac{c+a}{a+b}+\frac{a+b}{c+a}\right)}{\frac{c+a}{a+b}+\frac{a+b}{c+a}} \Rightarrow \frac{w_b}{w_c}+\frac{w_c}{w_b} \overset{③}{\geq} \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \text{ and so, ①, ② and ③}$$

$$\Rightarrow max\left\{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}, \frac{a+b}{a+c}+\frac{a+c}{a+b}\right\} \leq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \leq \frac{w_b}{w_c}+\frac{w_c}{w_b} \; \forall \Delta ABC,$$

$$'' ='' \text{ iff } b = c$$