

# ROMANIAN MATHEMATICAL MAGAZINE

In any  $\Delta ABC$  the following relationship holds :

$$s^2 + 16Rr + 22r^2 \leq (\sqrt{r_a r_b} + \sqrt{r_b r_c} + \sqrt{r_c r_a})^2 \leq \frac{1}{9}(18s^2 + 82Rr + 79r^2)$$

Proposed by Dang Ngoc Minh-Vietnam

**Solution by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \left(\sum_{\text{cyc}} \sqrt{r_b r_c}\right)^2 &= s^2 + 2s \cdot \sum_{\text{cyc}} \sqrt{(s-b)(s-c)} \stackrel{?}{\leq} \frac{1}{9}(18s^2 + 82Rr + 79r^2) \\ &\Leftrightarrow 2s \cdot \sum_{\text{cyc}} \sqrt{(s-b)(s-c)} \stackrel{?}{\leq} \frac{9s^2 + 82Rr + 79r^2}{9} \\ &\Leftrightarrow 2 \left(\sum_{\text{cyc}} x^2\right) \left(\sum_{\text{cyc}} yz\right) \stackrel{?}{\leq} \left(\sum_{\text{cyc}} x^2\right)^2 + \frac{82}{9} \cdot \frac{1}{4 \sum_{\text{cyc}} x^2} \cdot \prod_{\text{cyc}} (y^2 + z^2) + \frac{79}{9} \cdot \frac{x^2 y^2 z^2}{\sum_{\text{cyc}} x^2} \\ &\left(x = \sqrt{s-a}, y = \sqrt{s-b}, z = \sqrt{s-c} \Rightarrow s = \sum_{\text{cyc}} x^2\right) \Leftrightarrow 18 \left(\sum_{\text{cyc}} x^2\right)^3 + \\ &\quad 41 \left(\sum_{\text{cyc}} x^2\right) \left(\sum_{\text{cyc}} x^2 y^2\right) + 117 x^2 y^2 z^2 \stackrel{?}{\leq} 36 \left(\sum_{\text{cyc}} x^2\right) \left(\sum_{\text{cyc}} xy\right) \end{aligned}$$

Assigning  $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$  and  $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$  form sides of a triangle with semiperimeter, circumradius & inradius  
 $= s_0, R_0, r_0$  (say)

and then :  $\sum_{\text{cyc}} x = s_0, xyz = r_0^2 s_0, \sum_{\text{cyc}} xy = 4R_0 r_0 + r_0^2, \sum_{\text{cyc}} x^2 = s_0^2 - 8R_0 r_0 - 2r_0^2,$

$\sum_{\text{cyc}} x^2 y^2 = r_0^2 ((4R_0 + r_0)^2 - 2s_0^2)$ , & then, following simplification, (\*) becomes :

$$9s_0^6 - (288R_0 r_0 + 133r_0^2)s_0^4 + r_0^2(3208R_0^2 + 1932R_0 r_0 + 341r_0^2)s_0^2 - 185r_0^3(4R_0 + r_0)^3 \stackrel{?}{\leq} 0$$

and we now prove :  $9s^6 - (288Rr + 133r^2)s^4 +$

$$r^2(3208R^2 + 1932Rr + 341r^2)s^2 - 185r^3(4R + r)^3 \stackrel{?}{\leq} 0 \quad \text{①} \quad \& \because 9(s^2 - 16Rr + 5r^2)^3$$

Gerretsen  $\geq 0 \therefore$  to prove ①, it suffices to prove : LHS of ①  $\stackrel{?}{\geq} (s^2 - 16Rr + 5r^2)^3$   
 $\Leftrightarrow (72R - 124r)s^4 - r(1852R^2 - 3126Rr + 167r^2)s^2 +$

$$r^2(12512R^3 - 21720R^2 r + 4290Rr^2 - 655r^3) \stackrel{?}{\leq} 0 \quad \text{②}$$

and again, since

$(72R - 124r)(s^2 - 16Rr + 5r^2)^2 \stackrel{\text{Gerretsen}}{\geq} 0 \therefore$  in order to prove ②,  
 it suffices to prove : LHS of ②  $\stackrel{?}{\geq} (72R - 124r)(s^2 - 16Rr + 5r^2)^2 \Leftrightarrow$   
 $(452R^2 - 1562Rr + 1073r^2)s^2 \stackrel{?}{\geq} r(5920R^3 - 21544R^2r + 17350Rr^2 - 2445r^3)$

**Case 1**  $452R^2 - 1562Rr + 1073r^2 \geq 0$  and then : LHS of ③  $\stackrel{\text{Gerretsen}}{\geq}$

$(452R^2 - 1562Rr + 1073r^2)(16Rr - 5r^2) \stackrel{?}{\geq}$  RHS of ③  $\Leftrightarrow$   
 $(t - 2)((t - 2)(328t - 115) + 135) \stackrel{?}{\geq} 0 \left( t = \frac{R}{r} \right) \rightarrow$  true  $\because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow$  ③ is true

**Case 2**  $452R^2 - 1562Rr + 1073r^2 < 0$  and then : LHS of ③  $\stackrel{\text{Rouche}}{\geq}$

$$(452R^2 - 1562Rr + 1073r^2) \left( 2R^2 + 10Rr - r^2 + 2(R - 2r) \cdot \sqrt{R^2 - 2Rr} \right) \stackrel{?}{\geq} \text{③}_{\text{RHS}}$$

$$\Leftrightarrow 2(R - 2r) \overbrace{\left( 452R^3 - 1358R^2r + 1093Rr^2 - 343r^3 \right)}^{\sigma > 0} \stackrel{?}{\geq}$$

$$2(R - 2r) \cdot \sqrt{R^2 - 2Rr} \cdot \overbrace{\left( -(452R^2 - 1562Rr + 1073r^2) \right)}^{\mu > 0}$$

$$\Leftrightarrow \sigma^2 \stackrel{?}{\geq} (R^2 - 2Rr)\mu^2 \left( \because R \stackrel{\text{Euler}}{\geq} 2r \right) \Leftrightarrow (t - 2)(32t^3 + 8t^2 + 36t - 28) + 25 +$$

$$64 \left( (9266t^2 + 2444t + 11177)(t - 2)^3 + 22343t^2 - 90307t + 91253 \right) \stackrel{?}{\geq} 0$$

$\rightarrow$  true (strict inequality)  $\because t \stackrel{\text{Euler}}{\geq} 2$  and  $\because (90307)^2 - 4(22343)(91253) < 0$   
 $\Rightarrow$  ③ is true and combining, ③  $\Rightarrow$  ②  $\Rightarrow$  ①  $\Rightarrow$  (\*)  $\Rightarrow$  (\*) is true and again,

$$\left( \sum_{\text{cyc}} \sqrt{r_b r_c} \right)^2 \stackrel{\text{GM-HM}}{\geq} s^2 + \sum_{\text{cyc}} \frac{4s(s-b)(s-c)}{s-b+s-c} = \sum_{\text{cyc}} \frac{4sbc(s-b)(s-c)}{4Rrs}$$

$$= s^2 + \frac{r(s^2 + (4R+r)^2)}{R} \stackrel{\text{Gerretsen} + \text{Euler}}{\geq} s^2 + \frac{r(14Rr - r^2 + (4R+r)^2)}{R}$$

$$= s^2 + 16Rr + 22r^2 \text{ and so, } s^2 + 16Rr + 22r^2 \leq \left( \sqrt{r_a r_b} + \sqrt{r_b r_c} + \sqrt{r_c r_a} \right)^2 \leq$$

$$\frac{1}{9} (18s^2 + 82Rr + 79r^2) \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$