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In any ΔABC , the following relationship holds :

$$\frac{\sqrt{s^2 + 37r^2}}{4r} \geq \frac{m_a}{h_a} + \frac{h_a}{m_a} \geq \max \left\{ \frac{w_a}{h_a} + \frac{h_a}{w_a}, \frac{m_a}{w_a} + \frac{w_a}{m_a} \right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \frac{\sqrt{s^2 + 37r^2}}{4r} &\geq \frac{m_a}{h_a} + \frac{h_a}{m_a} \Leftrightarrow \frac{s^2 + 37r^2}{16r^2} - 4 \geq \frac{m_a^2}{h_a^2} + \frac{h_a^2}{m_a^2} - 2 \\ &\Leftrightarrow \frac{s^2 - 27r^2}{16r^2} \geq \frac{\left((s(s-a) + \frac{(b-c)^2}{4}) - (s(s-a) - s(s-a) \cdot \frac{(b-c)^2}{a^2}) \right)^2}{h_a^2 m_a^2} \\ &\Leftrightarrow \frac{s^2 - 27r^2}{16r^2} \geq \frac{(b-c)^4 ((b+c)^2 - a^2 + a^2)^2}{16a^4 \cdot \frac{4r^2 s^2}{a^2} \cdot m_a^2} \Leftrightarrow s^2 - 27r^2 \stackrel{?}{\geq} \frac{(b-c)^4 (b+c)^4}{4a^2 m_a^2 s^2} \end{aligned}$$

and now, $4a^2 m_a^2 - (b^2 - c^2)^2 = a^2(2b^2 + 2c^2 - a^2) - b^4 - c^4 + 2b^2 c^2 =$
 $2 \sum_{cyc} a^2 b^2 - \sum_{cyc} a^4 = 16r^2 s^2 > 0 \Rightarrow 4a^2 m_a^2 > (b-c)^2 (b+c)^2$ and so,

in order to prove (*), it suffices to prove : $s^2 (s^2 - 27r^2) \stackrel{?}{\geq} (b^2 - c^2)^2$

$$\begin{aligned} &\Leftrightarrow (x+y+z)^4 - 27xyz(x+y+z) \stackrel{?}{\geq} ((z+x)^2 - (x+y)^2)^2 \\ &\quad \left(\begin{array}{l} x = s-a, y = s-b, z = s-c \Rightarrow \\ a = y+z, b = z+x, c = x+y \text{ and } s = x+y+z \end{array} \right) \end{aligned}$$

$$\Leftrightarrow x^4 + 4x^3(y+z) + 2x^2(y+z)^2 - 11x^2yz - 11xyz(y+z) + 4yz(y+z)^2 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow x^4 + 4x^3 \cdot mx + 2x^2(mx)^2 - 11x^2 \cdot nx^2 - 11x \cdot nx^2(mx) + 4nx^2(mx)^2 \stackrel{?}{\geq} 0$$

$$\left(m = \frac{y+z}{x}, n = \frac{yz}{x^2} \right) \Leftrightarrow 2m^2 + 4m + 1 + n(4m^2 - 11m - 11) \stackrel{?}{\geq} 0$$

and it's trivially true if : $4m^2 - 11m - 11 \geq 0$ and when : $4m^2 - 11m - 11 < 0$,

then, as $n \stackrel{AM-GM}{\leq} \frac{m^2}{4} \therefore$ in order to prove (**), it suffices to prove :

$$2m^2 + 4m + 1 + \frac{m^2}{4}(4m^2 - 11m - 11) \stackrel{?}{\geq} 0 \Leftrightarrow \frac{(m+1)(4m+1)(m-2)^2}{4} \stackrel{?}{\geq} 0$$

\rightarrow true $\because m > 0 \Rightarrow (**)$ $\Rightarrow (*)$ is true $\therefore \frac{m_a}{h_a} + \frac{h_a}{m_a} \geq \frac{\sqrt{s^2 + 37r^2}}{4r}$

$$\begin{aligned} \text{Now, } \frac{m_a}{h_a} + \frac{h_a}{m_a} &\stackrel{?}{\geq} \frac{w_a}{h_a} + \frac{h_a}{w_a} \Leftrightarrow \frac{m_a^2 + h_a^2}{m_a h_a} \stackrel{?}{\geq} \frac{w_a^2 + h_a^2}{w_a h_a} \Leftrightarrow \frac{m_a^2 + h_a^2}{w_a^2 + h_a^2} - 1 \stackrel{?}{\geq} \frac{m_a}{w_a} - 1 \\ &\Leftrightarrow \frac{(m_a - w_a)(m_a + w_a)}{w_a^2 + h_a^2} \stackrel{?}{\geq} \frac{m_a - w_a}{w_a} \Leftrightarrow w_a(m_a + w_a) \stackrel{?}{\geq} w_a^2 + h_a^2 \end{aligned}$$

$$(\because m_a - w_a \geq 0) \Leftrightarrow m_a w_a \stackrel{?}{\geq} h_a^2 \rightarrow \text{true} \because m_a, w_a \geq h_a \therefore \frac{m_a}{h_a} + \frac{h_a}{m_a} \stackrel{①}{\geq} \frac{w_a}{h_a} + \frac{h_a}{w_a}$$

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$$\text{Again, } \frac{m_a}{h_a} + \frac{h_a}{m_a} \stackrel{?}{\geq} \frac{m_a}{w_a} + \frac{w_a}{m_a} \Leftrightarrow \frac{m_a^2 + h_a^2}{m_a h_a} \stackrel{?}{\geq} \frac{m_a^2 + w_a^2}{w_a m_a} \Leftrightarrow \frac{w_a}{h_a} - 1 \stackrel{?}{\geq} \frac{m_a^2 + w_a^2}{m_a^2 + h_a^2} - 1$$

$$\Leftrightarrow \frac{w_a - h_a}{h_a} \stackrel{?}{\geq} \frac{(w_a - h_a)(w_a + h_a)}{m_a^2 + h_a^2} \Leftrightarrow m_a^2 + h_a^2 \stackrel{?}{\geq} h_a(w_a + h_a) \quad (\because w_a - h_a \geq 0)$$

$$\Leftrightarrow m_a^2 \stackrel{?}{\geq} h_a w_a \rightarrow \text{true} \because m_a \geq h_a, w_a \therefore \frac{m_a}{h_a} + \frac{h_a}{m_a} \stackrel{\textcircled{2}}{\geq} \frac{m_a}{w_a} + \frac{w_a}{m_a} \text{ and so,}$$

$$\textcircled{1} \text{ and } \textcircled{2} \Rightarrow \boxed{\frac{m_a}{h_a} + \frac{h_a}{m_a} \geq \max \left\{ \frac{w_a}{h_a} + \frac{h_a}{w_a}, \frac{m_a}{w_a} + \frac{w_a}{m_a} \right\}}$$

$$\therefore \frac{\sqrt{s^2 + 37r^2}}{4r} \geq \frac{m_a}{h_a} + \frac{h_a}{m_a} \geq \max \left\{ \frac{w_a}{h_a} + \frac{h_a}{w_a}, \frac{m_a}{w_a} + \frac{w_a}{m_a} \right\} \forall \Delta ABC,$$

" = " for upper bound iff $y = z \wedge y + z = 2x \Rightarrow$ " = " iff ΔABC is equilateral
and " = " for lower bound iff $b = c$ (QED)