ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC holds :

$$|r_b - r_c| \le 4\sqrt{R^2 - (\sqrt{2}+1)Rr + 2(\sqrt{2}-1)r^2}$$

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$$\text{Via } (r_b - r_c)^2 = \frac{s(s-a)}{(s-b)(s-c)}, R = \frac{abc}{4.\sqrt{s(s-a)(s-b)(s-c)}} \text{ and}$$

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}, (r_b - r_c)^2 \overset{?}{\le} R^2 - (\sqrt{2}+1)Rr + 2(\sqrt{2}-1)r^2$$

$$\Leftrightarrow \frac{16\big(sabc - 8s(s-a)(s-b)(s-c)\big)\big(sabc + 4s(s-a)(s-b)(s-c)\big)}{4s.4s.s(s-a)(s-b)(s-c)} -$$

$$\frac{s^2(s-a)^2(b-c)^2}{s(s-a)(s-b)(s-c)} \overset{?}{\ge} 16\sqrt{2}.\frac{F}{s}.\frac{sabc - 8s(s-a)(s-b)(s-c)}{4s.F}$$

$$\Leftrightarrow 16x^8 + 32x^7(y+z) - 256x^6.yz - 32x^5\Big((y+z)^3 - 3yz(y+z)\Big) +$$
$$+128x^5yz(y+z) - 12x^4(((y+z)^2 - 2yz)^2 - 2y^2z^2) + 248x^4.yz((y+z)^2 - 2yz)$$
$$-504x^4.y^2z^2 + 8x^3(y+z)(((y+z)^2 - 2yz)^2 - yz(y+z)^2 + y^2z^2) -$$
$$160x^3.yz\Big((y+z)^3 - 3yz(y+z)\Big) + 248x^3.y^2z^2(y+z) +$$
$$4x^2((y+z)^2 - 2yz)(((y+z)^2 - 2yz)^2 - 3y^2z^2) +$$
$$12x^2.yz(((y+z)^2 - 2yz)^2 - 2y^2z^2) - 68x^2.y^2z^2(y+z)^2 - 16x^2.y^3z^3 +$$
$$4x.yz(y+z)(((y+z)^2 - 2yz)^2 - yz(y+z)^2 + y^2z^2) +$$
$$12x.y^2z^2\Big((y+z)^3 - 3yz(y+z)\Big) + 16x.y^3z^3(y+z) +$$
$$y^2z^2(((y+z)^2 - 2yz)^2 - 2y^2z^2) + 4y^3z^3((y+z)^2 - 2yz) + 6y^4z^4 \overset{?}{\ge} 0$$
$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \Rightarrow \\ a = y+z, b = z+x, c = x+y \text{ and } s = x+y+z \end{matrix}\right)$$

$$\Leftrightarrow \Big(\overbrace{m^4 - 8m^3 - 80m^2 + 768m - 1024}^{\sigma_1}\Big)n^2 +$$

$$\Big(\overbrace{4m^5 - 12m^4 - 200m^3 + 296m^2 + 224m - 256}^{\sigma_2}\Big)n +$$

$$\overbrace{4m^6 + 8m^5 - 12m^4 - 32m^3 + 32m + 16}^{\sigma_3} \boxed{\underset{(*)}{\overset{?}{\ge}}} 0 \quad \left(m = \frac{y+z}{x}, n = \frac{yz}{x^2}\right)$$

$$\text{and } \because \text{LHS of } (*) = (m-8)^2(m^2 + 8(m-8) + 48)n^2 +$$
$$(m-8)\Big((m-8)(4m^3 + 52m^2 + 376m + 2984) + 23904\Big)n +$$
$$4\Big((m-8)(m^5 + 10m^4 + 77m^3 + 608m^2 + 4864m + 38920) + 311364\Big) > 0$$

whenever $m \ge 8 \therefore$ we now focus on the case when : $m < 8$

$\boxed{Case\ 1}$ $m^4 - 8m^3 - 80m^2 + 768m - 1024 > 0$ (*and* $m < 8$) *and* then,

$\because \delta = \sigma_2^2 - 4\sigma_1\sigma_3 = 128(m+1)^2(m-2)^4(m-8)^2 \ge 0 \therefore$ in order to prove $(*)$, it suffices to prove : $2\sigma_1.n \overset{?}{\le} -\sigma_2 - \sqrt{\delta}$ and $\because n \overset{AM-GM}{\le} \frac{m^2}{4}$ $\therefore$ it suffices to prove :

$$2\sqrt{\delta} \overset{?}{\le} -2\sigma_2 - m^2\sigma_1 = (m-2)^2\overbrace{(8-m)}^{>0}(m^3+12m^2+4m+16)$$

$$\Leftrightarrow (m-2)^4(m-8)^2(m^3+12m^2+4m+16) \overset{?}{\ge} 512(m+1)^2(m-2)^4(m-8)^2$$

$$\Leftrightarrow (m-2)^4(m-8)^2(m^6+24m^5+152m^4+128m^3-112m^2-896m-256)$$

$$\overset{?}{\ge} 0 \Leftrightarrow (m-2)^4(m-8)^2.\frac{\begin{pmatrix} m^4-8m^3-80m^2+ \\ 768m-1024 \end{pmatrix}\begin{pmatrix} m^4+16m^3+ \\ 40m^2+64m+16 \end{pmatrix}}{(m-8)^2} \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because m^4-8m^3-80m^2+768m-1024 > 0 \text{ and } m < 8$$

Case 2 $\sigma_1 = m^4-8m^3-80m^2+768m-1024 < 0$ (and $m < 8$) and then :

$(*) \Leftrightarrow (-\sigma_1)n^2 - \sigma_2 n - \sigma_3 \underbrace{\overset{?}{\ge}}_{(**)} 0$ and then $\because$ discriminiant $= \sigma_2^2 - 4\sigma_1\sigma_3 = \delta \ge 0$

$\therefore$ to prove $(**)$, suffices to prove : $-2\sigma_1.n \underbrace{\overset{?}{\le}}_{①} \sigma_2 + \sqrt{\delta}$ AND $-2\sigma_1.n \underbrace{\overset{?}{\ge}}_{②} \sigma_2 - \sqrt{\delta}$

$\because -\sigma_1 > 0$ and $n \le \frac{m^2}{4}$ $\therefore$ to prove ①, it suffices to prove : $-2\sigma_2 - m^2\sigma_1 \overset{?}{\le} 2\sqrt{\delta}$

$$\Leftrightarrow (m-2)^2\overbrace{(8-m)}^{>0}(m^3+12m^2+4m+16) \overset{?}{\le} 2\sqrt{\delta} \Leftrightarrow$$

$$(m-2)^4(m-8)^2(m^3+12m^2+4m+16)^2 \overset{?}{\le} 512(m+1)^2(m-2)^4(m-8)^2$$

$$\Leftrightarrow (m-2)^4(m-8)^2.\frac{\begin{pmatrix} m^4-8m^3-80m^2+ \\ 768m-1024 \end{pmatrix}\begin{pmatrix} m^4+16m^3+ \\ 40m^2+64m+16 \end{pmatrix}}{(m-8)^2} \overset{?}{\le} 0 \rightarrow \text{true}$$

$\because m^4-8m^3-80m^2+768m-1024 < 0$ and $m < 8 \Rightarrow$ ① is true

and again, since : $-2\sigma_1.n > 0$ $\therefore$ in order to prove ②, it suffices to prove :

$\sqrt{\delta} \overset{?}{\ge} \sigma_2$ and it's trivially true if $\sigma_2 < 0$ and when : $\sigma_2 \ge 0$, it suffices to prove :

$$128(m+1)^2(m-2)^4(m-8)^2 \overset{?}{\ge} \begin{pmatrix} 4m^5-12m^4-200m^3+296m^2+ \\ 224m-256 \end{pmatrix}^2$$

$$\Leftrightarrow -16(m+1)^2(m-8)^2(m^6+8m^5-20m^4-32m^3+68m^2+32m-64) \overset{?}{\ge} 0$$

$$\Leftrightarrow -16(m+1)^2(m-8)^2.\frac{(m^4-8m^3-80m^2+768m-1024)(m^2-2)^2}{(m-8)^2} \overset{?}{\ge} 0$$

$\rightarrow$ true $\because m^4-8m^3-80m^2+768m-1024 < 0$ and $m < 8 \Rightarrow$ ② is true

$\therefore$ combining cases ① and ②, $(*)$ is true $\forall\, m, n > 0 \mid m^2 \ge 4n$ and so,

$$|r_b - r_c| \le 4.\sqrt{R^2 - (\sqrt{2}+1)Rr + 2(\sqrt{2}-1)r^2}\ \forall\, \Delta ABC,$$

" = " iff $y = z$ and $y + z = 2x \Rightarrow$ " = " iff Δ ABC is equilateral (QED)