ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\max\{h_a, h_b, h_c\} - \min\{h_a, h_b, h_c\} \geq \frac{s^2 - 12Rr - 3r^2}{2R}$$

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$$(\max\{h_a, h_b, h_c\} - \min\{h_a, h_b, h_c\})^2 = \left(\sum_{cyc} \frac{|h_b - h_c|}{2}\right)^2$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 45; published at www.ssmrmh.ro)

$$= \frac{1}{4}\sum_{cyc}(h_b - h_c)^2 + \frac{2}{4}\sum_{cyc}(|h_b - h_c||h_c - h_a|) \overset{\text{Triangle Inequality}}{\geq}$$

$$\frac{1}{2}\left(\sum_{cyc} h_a^2 - \sum_{cyc} h_a h_b\right) + \frac{1}{2}\cdot\left|\sum_{cyc}((h_b - h_c)(h_c - h_a))\right|$$

$$= \frac{1}{2}\left(\sum_{cyc} h_a^2 - \sum_{cyc} h_a h_b\right) + \frac{1}{2}\cdot\left|\sum_{cyc} h_b h_c - \sum_{cyc} h_a h_b - \sum_{cyc} h_c^2 + \sum_{cyc} h_c h_a\right|$$

$$= \frac{1}{2}\left(\sum_{cyc} h_a^2 - \sum_{cyc} h_a h_b\right) + \frac{1}{2}\left(\sum_{cyc} h_a^2 - \sum_{cyc} h_a h_b\right) \left(\because \sum_{cyc} h_a^2 \geq \sum_{cyc} h_a h_b\right)$$

$$= \frac{(s^2 + 4Rr + r^2)^2 - 16Rrs^2 - 8Rrs^2}{4R^2} \overset{?}{\geq} \frac{(s^2 - 12Rr - 3r^2)^2}{4R^2}$$

$$\Leftrightarrow (R + r)s^2 \underset{(*)}{\overset{?}{\geq}} r(4R + r)^2$$

$$\text{Now, } (R + r)s^2 \overset{\text{Gerretsen}}{\geq} (R + r)(16Rr - 5r^2) \overset{?}{\geq} r(4R + r)^2 \Leftrightarrow 3r^2(R - 2r) \overset{?}{\geq} 0$$

$$\to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (*) \text{ is true} \therefore \max\{h_a, h_b, h_c\} - \min\{h_a, h_b, h_c\} \geq$$

$$\frac{s^2 - 12Rr - 3r^2}{2R} \; \forall \, \Delta \, ABC, '' ='' \text{ iff } \Delta \, ABC \text{ is equilateral (QED)}$$