ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\max\{r_a, r_b, r_c\} - \min\{r_a, r_b, r_c\} \ge 2\sqrt{R^2 - Rr - 2r^2}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$(\max\{r_a, r_b, r_c\} - \min\{r_a, r_b, r_c\})^2 = \left(\sum_{cyc} \frac{|r_b - r_c|}{2}\right)^2$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba − 45; published at www.ssmrmh.ro)

$$= \frac{1}{4}\sum_{cyc}(r_b - r_c)^2 + \frac{2}{4}\sum_{cyc}(|r_b - r_c||r_c - r_a|) \overset{\text{Triangle Inequality}}{\ge}$$

$$\frac{1}{2}\left(\sum_{cyc} r_a^2 - \sum_{cyc} r_a r_b\right) + \frac{1}{2}\cdot\left|\sum_{cyc}\left((r_b - r_c)(r_c - r_a)\right)\right|$$

$$= \frac{1}{2}\left(\sum_{cyc} r_a^2 - \sum_{cyc} r_a r_b\right) + \frac{1}{2}\cdot\left|\sum_{cyc} r_b r_c - \sum_{cyc} r_a r_b - \sum_{cyc} r_c^2 + \sum_{cyc} r_c r_a\right|$$

$$= \frac{1}{2}\left(\sum_{cyc} r_a^2 - \sum_{cyc} r_a r_b\right) + \frac{1}{2}\left(\sum_{cyc} r_a^2 - \sum_{cyc} r_a r_b\right)\left(\because \sum_{cyc} r_a^2 \ge \sum_{cyc} r_a r_b\right)$$

$$= (4R + r)^2 - 3s^2 \overset{\text{Gerretsen}}{\ge} (4R + r)^2 - 3(4R^2 + 4Rr + 3r^2) = 4(R^2 - Rr - 2r^2)$$

$$\therefore \max\{r_a, r_b, r_c\} - \min\{r_a, r_b, r_c\} \ge 2\sqrt{R^2 - Rr - 2r^2}\ \forall\ \Delta\ ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED)