ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\max\{a,b,c\}-\min\{a,b,c\}\geq\frac{\sqrt{s^2-27r^2}}{2}$$

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$$(\max\{a,b,c\}-\min\{a,b,c\})^2=\left(\sum_{cyc}\frac{|b-c|}{2}\right)^2$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 45; published at www.ssmrmh.ro)

$$=\frac{1}{4}\sum_{cyc}(b-c)^2+\frac{2}{4}\sum_{cyc}(|b-c||c-a|)\overset{\text{Triangle Inequality}}{\geq}$$

$$\frac{1}{2}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+\frac{1}{2}\cdot\left|\sum_{cyc}((b-c)(c-a))\right|$$

$$=\frac{1}{2}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+\frac{1}{2}\cdot\left|\sum_{cyc}bc-\sum_{cyc}ab-\sum_{cyc}c^2+\sum_{cyc}ca\right|$$

$$=\frac{1}{2}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+\frac{1}{2}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)\left(\because\sum_{cyc}a^2\geq\sum_{cyc}ab\right)$$

$$=2(s^2-4Rr-r^2)-(s^2+4Rr+r^2)=s^2-12Rr-3r^2\overset{?}{\geq}\frac{s^2-27r^2}{4}$$

$$\Leftrightarrow s^2\overset{?}{\geq}16Rr-5r^2\rightarrow\text{true via Gerretsen}$$

$$\therefore\max\{a,b,c\}-\min\{a,b,c\}\geq\frac{\sqrt{s^2-27r^2}}{2}\ \forall\,\Delta\,ABC,$$

" = " iff Δ ABC is equilateral (QED)