ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\le\sqrt{\frac{2R}{r}}$$

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$$let\ x=s-a, y=s-b, z=s-c\ then\ a=y+z, b=x+z, c=x+y\ \&\ s=x+y+z$$

$$\frac{h_a}{r_a}+\frac{r_a}{h_a}=\frac{2(s-a)}{a}+\frac{a}{2(s-a)}=\frac{2x}{y+z}+\frac{y+z}{2x}$$

$$\frac{R}{r}=\frac{\frac{abc}{4F}}{\frac{F}{s}}=\frac{sabc}{4F^2}=\frac{(x+y)(y+z)(z+x)}{4xyz}\ then\ \frac{2R}{r}=\frac{(x+y)(y+z)(z+x)}{2xyz}$$

$$We\ need\ to\ show:$$

$$\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\le\sqrt{\frac{2R}{r}}\ or,\ \frac{h_a}{r_a}+\frac{r_a}{h_a}+2\overset{squaring}{\le}\frac{2R}{r}$$

$$\frac{2x}{y+z}+\frac{y+z}{2x}+2\le\frac{(x+y)(y+z)(z+x)}{2xyz}$$

$$2xyz\left(\frac{2x}{y+z}+\frac{y+z}{2x}+2\right)\le(x+y)(y+z)(z+x)$$

$$R.H.S=(x+y)(y+z)(z+x)=x^2y+xy^2+y^2z+yz^2+z^2x+zx^2+2xyz$$

$$L.H.S=2xyz\left(\frac{2x}{y+z}+\frac{y+z}{2x}+2\right)=\frac{4x^2yz}{y+z}+y^2z+yz^2+4xyz$$

$$L.H.S-R.H.S=x^2y+xy^2+z^2x+zx^2-\frac{4x^2yz}{y+z}-2xyz$$

$$=x^2(y+z)+x(y^2+z^2)-2xyz-\frac{4x^2yz}{y+z}=x^2(y+z)+x(y+z)^2-4xyz-\frac{4x^2yz}{y+z}$$

$$=x(y+z)(x+y+z)-4xyz\left(\frac{x+y+z}{y+z}\right)=x(x+y+z)\left((y+z)-\frac{4yz}{y+z}\right)$$

$$=\frac{x(x+y+z)((y+z)^2-4yz)}{y+z}=\frac{x(x+y+z)(y-z)^2}{y+z}\ge 0$$

Equality holds for b=c.