ROMANIAN MATHEMATICAL MAGAZINE

$$In\ any\ \Delta ABC\ , a = \max\{a, b, c\}\ prove\ that:$$

$$\frac{r_b}{r_c} + \frac{r_c}{r_b} \leq \frac{s^2}{9r^2} - 1$$

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$$Let\ s - a = x, s - b = y, s - c = z\ then\ x + y + z = s\ and$$
$$F = Area = \sqrt{(x + y + z)xyz}$$
$$a = y + z, b = x + z, c = x + y$$

$$\frac{s^2}{9r^2} = \frac{s^4}{9F^2} = \frac{(x + y + z)^3}{9xyz}\ \&\ \frac{r_b}{r_c} + \frac{r_c}{r_b} = \frac{s - c}{s - b} + \frac{s - b}{s - c} = \frac{y}{z} + \frac{z}{y} = \frac{y^2 + z^2}{yz}$$

$$We\ need\ to\ show:$$
$$\frac{y^2 + z^2}{yz} \leq \frac{(x + y + z)^3}{9xyz} - 1\ or, (x + y + z)^3 \geq 9x(y^2 + yz + z^2)\ (1)$$
$$as\ a = \max\{a, b, c\}\ then\ y + z \geq x + y\ \&\ y + z \geq z + x\ or, x \leq y, z$$
$$let\ y = x + a\ \&\ z = x + b\ where\ a, b \geq 0$$

$$From\ (1)\ we\ need\ to\ show:$$

$$(3x + a + b)^3 \geq 9x((x + a)^2 + (x + a)(x + b) + (x + b)^2)$$
$$L.H.S = (3x)^3 + 3(3x)^2(a + b) + 3.(3x)(a + b)^2 + (a + b)^3$$
$$R.H.S = 9x(3x^2 + 3x(a + b) + (a^2 + ab + b^2)$$
$$L.H.S - R.H.S = 9xab + (a + b)^3 \geq 0\ as\ a, b \geq 0$$

$$(3x + a + b)^3 \geq 9x((x + a)^2 + (x + a)(x + b) + (x + b)^2)$$

$$Equality\ occurs\ when\ a = b = 0\ or\ x = y = z\ or\ a = b = c.$$