In any Δ ABC the following relationship holds :

$$|b-c| \le \min\left\{\sqrt{s^2-8Rr-11r^2}, \frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}\right\}$$

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$$|b-c| \overset{?}{\le} \sqrt{s^2-8Rr-11r^2}$$

$$\Leftrightarrow (y-z)^2 \overset{?}{\le} (x+y+z)^2 - \frac{8(y+z)(z+x)(x+y)}{4(x+y+z)} - \frac{11xyz}{x+y+z}$$

$$\left(\begin{matrix} x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y \text{ and} \\ s=x+y+z \end{matrix}\right)$$

$$\Leftrightarrow x^3+(y+z)x^2-7xyz+2yz(y+z) \overset{?}{\ge} 0$$

$$\Leftrightarrow x^3+(mx)x^2-7x.nx^2+2nx^2(mx) \overset{?}{\ge} 0 \left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right)$$

$$\Leftrightarrow 1+m+n(2m-7) \boxed{\underset{(*)}{\overset{?}{\ge}}} 0 \text{ and it's trivially true if : } 2m-7 \ge 0 \text{ and when :}$$

$$2m-7<0, \textit{then} : \text{since } n \overset{AM-GM}{\le} \frac{m^2}{4} \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : } 1+m+\frac{m^2}{4}.(2m-7) \overset{?}{\ge} 0 \Leftrightarrow (2m+1)(m-2)^2 \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because m>0 \text{ as } x,y,z>0 \Rightarrow (*) \text{ is true } \therefore |b-c| \le \sqrt{s^2-8Rr-11r^2} \rightarrow (1)$$

$$\text{and again, } |b-c| \overset{?}{\le} \frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}$$

$$\Leftrightarrow 16(y-z)^2 \overset{?}{\le} 32(x+y+z)^2 - \frac{437(y+z)(z+x)(x+y)}{4(x+y+z)} + \frac{10xyz}{x+y+z}$$

$$\Leftrightarrow 128x^3-53(y+z)x^2-117x((y+z)^2-2yz)+62xyz+$$
$$64\left((y+z)^3-3yz(y+z)\right)+11yz(y+z) \overset{?}{\ge} 0$$

$$\Leftrightarrow 128x^3-53(mx)x^2-117x((mx)^2-2nx^2)+62x.nx^2+$$
$$64\left((mx)^3-3nx^2(mx)\right)+11nx^2(mx) \overset{?}{\ge} 0$$

$$\Leftrightarrow 64m^3-117m^2-53m+128+n(296-181m) \boxed{\underset{(**)}{\overset{?}{\ge}}} 0$$

We shall now prove that : $64m^3-117m^2-53m+128>0 \; \forall \, m>0 \rightarrow$ (i)

Case 1 $m \le \frac{761}{535}$ and then : $64m^3-117m^2-53m+128$

$$= \frac{1}{125}\left((320m+311)(5m-7)^2+761-535m\right)>0$$

(inequality is strict because "m" cannot be simultaneously equal to $\frac{7}{5}$ and $\frac{761}{535}$)

Case 2 $m > \frac{761}{535}$ and then : $64m^3-117m^2-53m+128$

$$=\frac{1}{729}\Big((576m+611)(9m-13)^2+6993m-9947\Big)>0$$

$$\left(\because m>\frac{761}{535}>\frac{9947}{6993}\ as\ (761)(6993)-(535)(9947)=28>0\right)\ and\ so,$$

combining both cases, (i) is true and hence, (∗∗) it's trivially true if :

$$296-181m\geq 0\ \ and\ when: 296-181m<0, then: \text{since } n\overset{AM-GM}{\leq}\frac{m^2}{4}$$

∴ in order to prove (∗∗), it suffices to prove :

$$64m^3-117m^2-53m+128+\frac{m^2}{4}.(296-181m)\overset{?}{\geq}0$$

$$\Leftrightarrow (75m+128)(m-2)^2\overset{?}{\geq}0\rightarrow \text{true} \because m>0\ as\ x,y,z>0\Rightarrow (**)\text{ is true}$$

$$\therefore |b-c|\leq\frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}\rightarrow (2)\ \therefore (1)\ and\ (2)\Rightarrow$$

$$|b-c|\leq\sqrt{s^2-8Rr-11r^2},\frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}$$

$$\Rightarrow |b-c|\leq\min\left\{\sqrt{s^2-8Rr-11r^2},\frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}\right\}$$

$\forall\ \Delta ABC$, " = " iff $y=z$ and $y+z=2x\Rightarrow$ " = " iff ΔABC is equilateral (QED)