ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $a = \max\{a, b, c\}$ the following relationship holds :

$$\frac{b+c}{2a} + \frac{2a}{b+c} \le \frac{R}{2r} + 1$$

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$$\frac{b+c}{2a} + \frac{2a}{b+c} \overset{?}{\le} \frac{R}{2r} + 1$$

$$\Leftrightarrow \frac{(b+c)^2 + 4a^2 - 2a(b+c)}{2a(b+c)} \overset{?}{\le} \frac{abc}{8(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow (y+z)^2(z+x)(x+y)(2x+y+z) \overset{?}{\ge}$$
$$4xyz\left((2x+y+z)^2 + 4(y+z)^2 - 2(y+z)(2x+y+z)\right)$$

$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \text{ and} \\ s = x+y+z \end{matrix}\right)$$

$$\Leftrightarrow 2x^3(y+z)^2 - 16x^3.yz + 3x^2\left((y+z)^3 - 3yz(y+z)\right) + 9x^2.yz(y+z) +$$
$$x((y+z)^2 - 2yz)^2 - 6x.yz(y+z)^2 - 4x.y^2z^2 + yz(y+z)^3 \overset{?}{\ge} 0$$

$$\Leftrightarrow 2x^3.(mx)^2 - 16x^3.nx^2 + 3x^2((mx)^3 - 3.nx^2.mx) + 9x^2.nx^2.mx +$$
$$x((mx)^2 - 2nx^2)^2 - 6x.nx^2.(mx)^2 - 4x.n^2x^4 + nx^2.(mx)^3 \overset{?}{\ge} 0 \left(\begin{matrix} m = \frac{y+z}{x}, \\ n = \frac{yz}{x^2} \end{matrix}\right)$$

$$\Leftrightarrow m^4 + 3m^3 + 2m^2 + n(m^3 - 10m^2 - 16) \boxed{\overset{?}{\ge}_{(*)}} 0 \text{ and it's trivially true if :}$$

$$m^3 - 10m^2 - 16 \ge 0 \text{ and when : } m^3 - 10m^2 - 16 < 0, \text{then : since } n \overset{AM-GM}{\le} \frac{m^2}{4}$$

$\therefore$ in order to prove $(*)$, it suffices to prove :

$$m^4 + 3m^3 + 2m^2 + \frac{m^2}{4}.(m^3 - 10m^2 - 16) \overset{?}{\ge} 0 \Leftrightarrow m^2(m-2)^3 \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because m \ge 2 \text{ as } a \ge b, c \Rightarrow y, z \ge x \Rightarrow m = \frac{y+z}{x} \ge 2 \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{b+c}{2a} + \frac{2a}{b+c} \le \frac{R}{2r} + 1 \ \forall \Delta \text{ ABC}, '' ='' \text{ iff } y = z \text{ and } m = 2 \Rightarrow y + z = 2x \Rightarrow$$
$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$