ROMANIAN MATHEMATICAL MAGAZINE

In any acute Δ ABC the following relationship holds :

$$p_a p_b p_c \ge \frac{(R+4r)s^2}{6}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$p_a = \frac{2s}{2s+a}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 68; relation (•••); published at www.ssmrmh.ro)

$$\Rightarrow p_a \overset{Walker}{\ge} \frac{2s}{2s+a}\cdot\sqrt{2R^2+8Rr-16Rr\sin^2\frac{A}{2}} \text{ and analogs}$$

$$\Rightarrow p_a p_b p_c \ge \frac{8s^3}{2s(9s^2+6Rr+r^2)}\cdot\sqrt{\prod_{cyc}\left(2R^2+8Rr-16Rr\sin^2\frac{A}{2}\right)} \to (1)$$

$$\text{Now, }\prod_{cyc}\left(2R^2+8Rr-16Rr\sin^2\frac{A}{2}\right) =$$

$$(2R^2+8Rr)^3-16Rr(2R^2+8Rr)^2.\frac{2R-r}{2R}+$$

$$256R^2r^2(2R^2+8Rr)\left(\frac{(2R-r)^2}{4R^2}-\frac{2}{16R^2}\sum_{cyc}AI^2\right)-$$

$$(16)(256)R^3r^3.\frac{r^2}{16R^2}\left(\because \sum_{cyc}\sin^4\frac{A}{2} = \Bigg(\underbrace{\sum_{cyc}\sin^2\frac{A}{2}}_{\frac{2R-r}{2R}}\Bigg)^2 - \frac{2r^2}{16R^2}\operatorname{cosec}^2\frac{A}{2}\right)$$

$$= (2R^2+8Rr)^3-8r(2R-r)(2R^2+8Rr)^2+$$

$$32r^2(2R^2+8Rr)\left(2(2R-r)^2-(s^2-8Rr+r^2)\right)-256Rr^5 \overset{Gerretsen}{\ge}$$

$$(2R^2+8Rr)^3-8r(2R-r)(2R^2+8Rr)^2+$$

$$32r^2(2R^2+8Rr)\left(2(2R-r)^2-(4R^2-4Rr+4r^2)\right)-256Rr^5$$

$$\overset{\text{via (1) and Gerretsen}}{\Rightarrow} p_a p_b p_c \ge$$

$$\frac{4s^2}{9(4R^2+4Rr+3r^2)+6Rr+r^2}\cdot\sqrt{\begin{array}{c}(2R^2+8Rr)^3-8r(2R-r)(2R^2+8Rr)^2+\\32r^2(2R^2+8Rr)\left(2(2R-r)^2-(4R^2-4Rr+4r^2)\right)-\\256Rr^5\end{array}}$$

$$\overset{?}{\geq} \frac{(\mathbf{R}+4\mathbf{r})\mathbf{s}^2}{6} \overset{\text{squaring and simplifying}}{\Leftrightarrow}$$

$$828t^6 + 1260t^5 + 10863t^4 + 53484t^3 - 112180t^2 - 121568t - 3136 \overset{?}{\geq} 0$$

$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\left(828t^5 + 2916t^4 + 16695t^3 + 86874t^2 + 61568t + 1568\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore p_a p_b p_c \geq \frac{(R+4r)s^2}{6} \ \forall \text{ acute } ABC,$$

$$'' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$