

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$p_a \leq \frac{\sqrt{9s^2 - 48Rr - 66r^2}}{9r} \cdot h_a$$

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$$p_a^2 \stackrel{?}{\leq} \frac{9s^2 - 48Rr - 66r^2}{81r^2} \cdot h_a^2 \quad \Leftrightarrow$$

$$s(s-a) + \frac{s(3s+a)}{(2s+a)^2} \cdot (b-c)^2 \stackrel{?}{\leq}$$

$$\left(\frac{9s^3}{81(s-a)(s-b)(s-c)} - \frac{48abc}{81 \cdot 4(s-a)(s-b)(s-c)} - \frac{66}{81} \right) \left(\frac{4s(s-a)(s-b)(s-c)}{a^2} \right)$$

$$\Leftrightarrow \frac{x(y+z+2(x+y+z))^2 + (y+z+3(x+y+z))(y-z)^2}{(y+z+2(x+y+z))^2}$$

$$\stackrel{?}{\leq} \frac{12(x+y+z)^3 - 16(y+z)(z+x)(x+y) - 88xyz}{27(y+z)^2}$$

$$(x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \text{ and } s = x+y+z)$$

$$\Leftrightarrow 12x^5 + 56x^4(y+z) + 80x^3(y+z)^2 - 88x^3 \cdot nx^2 +$$

$$36x^2 \left((mx)^3 - 3nx^2(mx) \right) - 172x^2 \cdot nx^2 \cdot mx - 165x \cdot nx^2 \cdot (mx)^2 +$$

$$72 \cdot nx^2 \cdot \left((mx)^3 - 3nx^2(mx) \right) + 216n^2x^4 \cdot mx \stackrel{?}{\geq} 0 \quad \left(m = \frac{y+z}{x}, n = \frac{yz}{x^2} \right)$$

$$\Leftrightarrow n(72m^3 - 165m^2 - 280m - 88) + 36m^3 + 80m^2 + 56m + 12 \stackrel{?}{\geq} 0 \text{ and it's } \boxed{(*)}$$

trivially true if : $72m^3 - 165m^2 - 280m - 88 \geq 0$ and when :

$$72m^3 - 165m^2 - 280m - 88 < 0, \text{ then : since } n \stackrel{\text{AM-GM}}{\leq} \frac{m^2}{4}$$

\therefore in order to prove (*), it suffices to prove :

$$\frac{m^2}{4} \cdot (72m^3 - 165m^2 - 280m - 88) + 36m^3 + 80m^2 + 56m + 12 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow (8m+3)(3m+2)^2(m-2)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true} \because m > 0 \text{ as } x, y, z > 0 \Rightarrow (*) \text{ is true}$$

$$\therefore p_a \leq \frac{\sqrt{9s^2 - 48Rr - 66r^2}}{9r} \cdot h_a'' ='' \text{ iff } y = z \text{ and } y + z = 2x \Rightarrow$$

'' ='' iff ΔABC is equilateral (QED)