ROMANIAN MATHEMATICAL MAGAZINE

In any Δ AB, the following relationship holds

$$m_a \le \frac{\sqrt{9s^2-106Rr+113r^2}}{12r}.w_a$$

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$$m_a^2 \overset{?}{\le} \frac{9s^2-106Rr+113r^2}{144r^2}.w_a^2 \Leftrightarrow s(s-a)+\frac{(b-c)^2}{4} \overset{?}{\le}$$
$$\left(\frac{9s^3}{144(s-a)(s-b)(s-c)}-\frac{106abc}{144.4(s-a)(s-b)(s-c)}+\frac{113}{144}\right).\frac{4bc.s(s-a)}{(b+c)^2}$$
$$\Leftrightarrow \frac{4x(x+y+z)+(y-z)^2}{4} \overset{?}{\le}$$
$$\frac{18(x+y+z)^3-53(y+z)(z+x)(x+y)+226xyz}{288xyz}.\frac{4(z+x)(x+y)x(x+y+z)}{(2x+y+z)^2}$$
$$\left(\begin{matrix}x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y \text{ and} \\ s=x+y+z\end{matrix}\right)$$
$$\Leftrightarrow 19x^6+37x^5(y+z)+21x^4(y+z)^2-44x^4yz+21x^3\left((y+z)^3-3yz(y+z)\right)$$
$$-95x^3yz(y+z)+37x^2(y^2+z^2)^2+42x^2y^2z^2-162x^2yz(y^2+z^2)+$$
$$18(y+z)(y^5+z^5)-88xyz\left((y+z)^3-3yz(y+z)\right)+107xy^2z^2(y+z)+$$
$$19y^2z^2(y+z)^2 \overset{?}{\ge} 0 \Leftrightarrow 18+37m+21m^2-44n+21(m^3-3mn)-$$
$$95mn+37(m^2-2n)^2+42n^2-162n(m^2-2n)+18m((m^2-2n)^2+n^2-nm^2)$$
$$-88n(m^3-3mn)+107n^2m+19n^2m^2 \overset{?}{\ge} 0 \left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right)$$
$$\Leftrightarrow (19m^2+461m+514)n^2-(178m^3+310m^2+158m+514+44)n+$$
$$18m^5+37m^4+21m^3+21m^2+37m+18 \boxed{\overset{?}{\underset{(*)}{\ge}}} 0$$

Now, LHS of $(*)$ is a quadratic polynomial in "n" with discriminant $=$

$$(178m^3+310m^2+158m+514+44)^2-$$
$$4(19m^2+461m+514)(18m^5+37m^4+21m^3+21m^2+37m+18)$$
$$=-4(m-2)^2(342m^5+2448m^4+7542m^3+11387m^2+8152m+2192) \le 0$$

$\because m>0$ as $x,y,z>0 \Rightarrow (*)$ is true $\therefore m_a \le \dfrac{\sqrt{9s^2-106Rr+113r^2}}{12r}.w_a \ \forall \Delta ABC$,

"=" iff $b+c=2a$ (QED)