

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$g_a \leq \sqrt{\frac{R + 2r}{4r}} \cdot h_a$$

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$$\begin{aligned}
 g_a^2 &\stackrel{?}{\leq} \left(\frac{R}{4r} + \frac{1}{2}\right) \cdot h_a^2 \stackrel{\text{via Bogdan Fustei}}{\Leftrightarrow} s(s-a) - \frac{s-a}{a} \cdot (b-c)^2 \stackrel{?}{\leq} \\
 &\left(\frac{abc}{16(s-a)(s-b)(s-c)} + \frac{1}{2}\right) \left(s(s-a) - \frac{s(s-a)}{a^2} \cdot (b-c)^2\right) \\
 \Leftrightarrow \frac{sa - (b-c)^2}{a} &\stackrel{?}{\leq} s \cdot \frac{abc + 8(s-a)(s-b)(s-c)}{16(s-a)(s-b)(s-c)} \cdot \frac{4(s-b)(s-c)}{a^2} \\
 \Leftrightarrow ((y+z)(z+x)(x+y) + 8xyz)(x+y+z) &\stackrel{?}{\geq} 4(y+z)x \left(\frac{(y+z)(x+y+z)}{(y-z)^2} - \right) \\
 (x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \text{ and}) & \\
 s = x+y+z & \\
 \Leftrightarrow x^3(y+z) - 2x^2(y+z)^2 + 8x^2 \cdot yz + x \left((y+z)^3 - 3yz(y+z) \right) - & \\
 4x \cdot yz(y+z) + yz(y+z)^2 &\stackrel{?}{\geq} 0 \\
 \Leftrightarrow x^3(mx) - 2x^2(mx)^2 + 8x^2 \cdot nx^2 + x \left((mx)^3 - 3nx^2(mx) \right) - & \\
 4x \cdot nx^2(mx) + nx^2(mx)^2 &\stackrel{?}{\geq} 0 \left(m = \frac{y+z}{x}, n = \frac{yz}{x^2} \right) \\
 \Leftrightarrow n(m^2 - 7m + 8) + m(m-1)^2 &\stackrel{?}{\geq} 0 \text{ and it's trivially true if :}
 \end{aligned}$$

$$m^2 - 7m + 8 \geq 0 \text{ and when : } m^2 - 7m + 8 < 0, \text{ then : since } n \stackrel{\text{AM-GM}}{\leq} \frac{m^2}{4}$$

$$\therefore \text{ in order to prove } (*), \text{ it suffices to prove : } \frac{m^2}{4} \cdot (m^2 - 7m + 8) + m(m-1)^2 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow (m+1)(m-2)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true} \because m > 0 \text{ as } x, y, z > 0 \Rightarrow (*) \text{ is true}$$

$$\begin{aligned}
 \therefore g_a &\leq \sqrt{\frac{R + 2r}{4r}} \cdot h_a, \text{ " = " iff } y = z \text{ and } y + z = 2x \Rightarrow \\
 \text{" = " iff } \Delta ABC &\text{ is equilateral (QED)}
 \end{aligned}$$