ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\le\frac{w_b}{w_c}+\frac{w_c}{w_b}\le\frac{r_b+r_c}{w_a}$$

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$$\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\overset{?}{\le}\frac{w_b}{w_c}+\frac{w_c}{w_b}\Leftrightarrow\frac{b}{c}+\frac{c}{b}+2\overset{?}{\le}\frac{w_b^2}{w_c^2}+\frac{w_c^2}{w_b^2}+2$$

$$\Leftrightarrow\frac{(b+c)^2}{bc}\overset{?}{\le}\frac{\left(sa-\frac{s(s-b)(c-a)^2}{(c+a)^2}-\frac{s(s-c)(a-b)^2}{(a+b)^2}\right)^2}{\frac{4ca}{(c+a)^2}.s(s-b).\frac{4ab}{(a+b)^2}.s(s-c)}$$

$$\Leftrightarrow\begin{pmatrix}(y+z)(2y+z+x)^2(2z+x+y)^2-y(z-x)^2(2z+x+y)^2-\\ z(x-y)^2(2y+z+x)^2\end{pmatrix}\overset{?}{\ge}$$

$$16(y+z)^2yz(2x+y+z)^2(2y+z+x)^2(2z+x+y)^2$$

$$\begin{pmatrix}x=s-a,y=s-b,z=s-c\Rightarrow\\ a=y+z,b=z+x,c=x+y\text{ and }s=x+y+z\end{pmatrix}\overset{\text{Following simplification}}{\Leftrightarrow}$$

$$x^6(y-z)^2(y+z)^2+6x^5(y-z)^2(y+z)\left((y^2+yz+z^2)+yz\right)+$$
$$x^4(y-z)^2(15(y^2+yz+z^2)^2+31yz(y^2+yz+z^2)+16y^2z^2)+$$
$$x^3(y-z)^2(y+z)\left(20(y^2+yz+z^2)(y^2+z^2)+24y^2z^2+64yz(y^2+yz+z^2)\right)+$$
$$x^2(y-z)^2(y+z)^2(15(y^2+z^2)^2+65yz(y^2+z^2)+69y^2z^2)+$$
$$x(y-z)^2(y+z)\begin{pmatrix}6(y^2+z^2)(y^4+y^3z+y^2z^2+yz^3+z^4)+\\ 32yz(y^2+z^2)(y^2+yz+z^2)+52y^2z^2(y^2+yz+z^2)\end{pmatrix}+$$
$$(y-z)^2\begin{pmatrix}6yz(y^2+yz+z^2)(y^4+y^3z+y^2z^2+yz^3+z^4)+\\ 13y^2z^2(y^2+yz+z^2)^2+\\ (y+z)(y^2+z^2)(y^2+yz+z^2)(y^3+z^3)+y^3z^3(y^2+yz+z^2)\end{pmatrix}\overset{?}{\ge}0$$

$$\rightarrow\text{true}\ \therefore\frac{w_b}{w_c}+\frac{w_c}{w_b}\ge\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\text{ and again, }\frac{w_b}{w_c}+\frac{w_c}{w_b}\overset{?}{\le}\frac{r_b+r_c}{w_a}$$

$$\Leftrightarrow\frac{\left(sa-\frac{s(s-b)(c-a)^2}{(c+a)^2}-\frac{s(s-c)(a-b)^2}{(a+b)^2}\right)^2}{\frac{4ca}{(c+a)^2}.s(s-b).\frac{4ab}{(a+b)^2}.s(s-c)}\overset{?}{\le}\frac{s(s-a)(s-b)(s-c)a^2}{(s-b)^2(s-c)^2.\frac{4bc}{(b+c)^2}.s(s-a)}\overset{\substack{\text{via earlier substitution}\\ \text{and following simplification}}}{\Leftrightarrow}$$

$$16x^5(y-z)^2(y+z)^3+x^4(y-z)^2(y+z)^2\left(152yz+68(y^2+z^2)\right)+$$
$$x^3(y-z)^2\begin{pmatrix}376yz(y^2+yz+z^2)(y+z)+\\ 104(y^2+yz+z^2)(y^2+z^2)(y+z)+168y^2z^2(y+z)\end{pmatrix}+$$

$$x^2(y-z)^2(y+z)^2\left(480y^2z^2+68(y^2+z^2)^2+360yz(y^2+z^2)\right)+$$

$$x(y-z)^2(y+z)\begin{pmatrix}16(y^4+y^3z+y^2z^2+yz^3+z^4)(y^2+z^2)+\\136yz(y^2+yz+z^2)(y^2+z^2)+\\344y^2z^2(y+z)(y^2+yz+z^2)+y^3z^3\end{pmatrix}+$$

$$(y-z)^2(y+z)^2\left(16yz(y^2+z^2)^2+88y^3z^3+68y^2z^2(y^2+z^2)\right)\overset{?}{\geq}0\to \text{true}$$

$$\therefore \frac{w_b}{w_c}+\frac{w_c}{w_b}\leq\frac{r_b+r_c}{w_a}\ \text{and so,}\ \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\leq\frac{w_b}{w_c}+\frac{w_c}{w_b}\leq\frac{r_b+r_c}{w_a}\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } b=c\ (QED)$$