

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$\frac{2(b^2 + bc + c^2)}{3(b + c)} \cos \frac{A}{2} \leq p_a \leq \frac{3\sqrt{3} \cdot (3b^2 - bc + 3c^2)}{20s}$$

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\Leftrightarrow

$$p_a^2 \stackrel{?}{\leq} \frac{27(3b^2 - bc + 3c^2)^2}{400s^2} \Leftrightarrow \frac{s(s-a)(2s+a)^2 + s(3s+a)(b-c)^2}{(2s+a)^2} \stackrel{?}{\leq} \frac{27(3b^2 - bc + 3c^2)^2}{400s^2}$$

$$\begin{aligned} &\Leftrightarrow 27(y+z+2(x+y+z))^2 \left(3(z+x)^2 + 3(x+y)^2 - (z+x)(x+y) \right)^2 \stackrel{?}{\geq} \\ &400(x+y+z)^3 \left(x(y+z+2(x+y+z))^2 + (y+z+3(x+y+z))(y-z)^2 \right) \\ &\left(\begin{array}{l} x = s - a, y = s - b, z = s - c \Rightarrow \\ a = y + z, b = z + x, c = x + y \text{ and } s = x + y + z \end{array} \right) \Leftrightarrow 1100x^6 + 3900x^5 \cdot mx + \\ &4215x^4 \cdot m^2x^2 - 2760x^4 \cdot nx^2 + 1210x^3(m^3x^3 - 3nx^2 \cdot mx) - 5810x^3 \cdot nx^2 \cdot mx \\ &+ 57x^2((m^2x^2 - 2nx^2)^2 - 2n^2x^4) - 10398x^2 \cdot nx^2 \cdot (m^2x^2 - 2nx^2) - \\ &15618x^2 \cdot n^2x^4 + 606x \cdot mx((m^2x^2 - 2nx^2)^2 + n^2x^4 - nx^2 \cdot m^2x^2) - \\ &3588x \cdot nx^2(m^3x^3 - 3nx^2 \cdot mx) + 2082x \cdot n^2x^4 \cdot mx + \\ &587(m^2x^2 - 2nx^2)((m^2x^2 - 2nx^2)^2 - 3n^2x^4) - \\ &284nx^2((m^2x^2 - 2nx^2)^2 - 2n^2x^4) + 5488n^2x^4 \cdot m^2x^2 + 1742n^3x^6 \stackrel{?}{\geq} 0 \\ &\left(m = \frac{y+z}{x}, n = \frac{yz}{x^2} \right) \Leftrightarrow (11907m^2 + 15876m + 5292)n^2 - \\ &(3806m^4 + 6618m^3 + 10626m^2 + 9440m + 2760)n + \\ &587m^6 + 606m^5 + 57m^4 + 1210m^3 + 4215m^2 + 3900m + 1100 \stackrel{?}{\geq} 0 \end{aligned}$$

Now, LHS of (*) is a quadratic expression in "n" with discriminant $\delta = (3806m^4 + 6618m^3 + 10626m^2 + 9440m + 2760)^2 -$

$$4(11907m^2 + 15876m + 5292) \left(\begin{array}{l} 587m^6 + 606m^5 + 57m^4 + 1210m^3 + \\ 4215m^2 + 3900m + 1100 \end{array} \right)$$

$$\equiv -6400(m+1)^3(2105m^5 - 3852m^4 - 5862m^3 + 5408m^2 + 8328m + 2448)$$

and if : $P = 2105m^5 - 3852m^4 - 5862m^3 + 5408m^2 + 8328m + 2448 \geq 0$,

then : $\delta \leq 0 \Rightarrow (*)$ is true and so, we now focus on the case when : $P < 0$

($\Rightarrow \delta > 0$) and then, in order to prove (*), it suffices to prove :

$$2(11907m^2 + 15876m + 5292)n \stackrel{?}{\leq} 3806m^4 + 6618m^3 + 10626m^2 +$$

$$9440m + 2760 - \sqrt{\delta} \text{ and } \therefore n \stackrel{AM-GM}{\leq} \frac{m^2}{4} \therefore \text{it suffices to prove :}$$

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$$2(11907m^2 + 15876m + 5292) \cdot \frac{m^2}{4} \stackrel{?}{\leq} 3806m^4 + 6618m^3 + 10626m^2 + 9440m$$

$$+ 2760 - \sqrt{\delta} \Leftrightarrow 2\sqrt{\delta} \stackrel{?}{\geq} 5(-859m^4 - 528m^3 + 3192m^2 + 3776m + 1104)$$

and let us **assume** $Q = 859m^4 + 528m^3 - 3192m^2 - 3776m - 1104 \geq 0$

whenever : $P < 0$ and then : $P < 0 \leq \frac{m}{5} \cdot Q \Rightarrow mQ > 5P \Rightarrow T$

$$= 1611m^5 - 3298m^4 - 4353m^3 + 5136m^2 + 7124m + 2040 \stackrel{①}{<} 0$$

Case 1 $m \geq 1$ and then : $T = (m - 1)(m - 2)^2(1611m^2 + 4757m + 6544) + 6244m^2 - 26200m + 28216 > 0$ $\because \Delta$ of $(6244m^2 - 26200m + 28216) = (26200)^2 - 4(6244)(28216) \ll 0 \therefore T > 0$

Case 2 $0 < m < 1$; then : $T = (m - 1)(m - 2)(4m - 1)^2 \left(\frac{3222m + 4681}{32} \right) + \frac{13024m + 3801m(1 - m^2) + 2578m(1 - m)}{2} + \frac{15m + 2m^3 + m^2 + 55918}{32}$

$> 0 \therefore$ combining both cases, $T > 0 \forall m > 0$ and this is a contradiction to ① and

hence our assumption is incorrect $\therefore Q < 0$ and so, $(**) \Leftrightarrow$

$$-25600(m + 1)^3(2105m^5 - 3852m^4 - 5862m^3 + 5408m^2 + 8328m + 2448) \stackrel{?}{\leq} 25(-859m^4 - 528m^3 + 3192m^2 + 3776m + 1104)^2 \Leftrightarrow$$

$$33075(3m + 2)^4(m - 2)^2(27m^2 + 68m + 44) \stackrel{?}{\geq} 0 \rightarrow \text{true} \Rightarrow (***) \Rightarrow (*) \text{ is true}$$

$$\therefore p_a \leq \frac{3\sqrt{3} \cdot (3b^2 - bc + 3c^2)}{20s} \text{ and again, } \frac{2(b^2 + bc + c^2)}{3(b + c)} \cos \frac{A}{2} \leq p_a \Leftrightarrow$$

$$\frac{s(s - a)(2s + a)^2 + s(3s + a)(b - c)^2}{(2s + a)^2} \stackrel{?}{\geq} \frac{4(b^2 + bc + c^2)^2}{9(b + c)^2} \cdot \frac{s(s - a)}{bc}$$

via earlier substitution
and following simplification

\Leftrightarrow

$$(y - z)^2 \left(\begin{aligned} &48x^5 + 120x^4(y + z) + 92x^3(y + z)^2 + 112x^3yz + \\ &24x^2(y + z)(y^2 + yz + z^2) + 288x^2yz(y + z) + 180xyz(y + z)^2 + \\ &36yz(y + z)(y^2 + yz + z^2) + 36y^2z^2(y + z) \end{aligned} \right) \stackrel{?}{\geq} 0$$

$$\rightarrow \text{true} \therefore p_a \geq \frac{2(b^2 + bc + c^2)}{3(b + c)} \cos \frac{A}{2} \text{ and so,}$$

$$\frac{2(b^2 + bc + c^2)}{3(b + c)} \cos \frac{A}{2} \leq p_a \leq \frac{3\sqrt{3} \cdot (3b^2 - bc + 3c^2)}{20s},$$

" = " for lower bound iff $b = c$ and

" = " for upper bound iff ΔABC is equilateral (QED)