ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$n_a \le \frac{3\sqrt{3}(5b^2-6bc+5c^2)}{16s}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$n_a^2 \overset{?}{\le} \frac{27(5b^2-6bc+5c^2)^2}{256s^2} \overset{\text{Bogdan Fustei}}{\Leftrightarrow} \frac{as(s-a)+s(b-c)^2}{a} \overset{?}{\le} \frac{27(5b^2-6bc+5c^2)^2}{256s^2}$$

$$\Leftrightarrow 27(y+z)\left(5(z+x)^2+5(x+y)^2-6(z+x)(x+y)\right)^2 \overset{?}{\ge} 256(x+y+z)^3(x(y+z)+(y-z)^2)$$

$$\left(\text{denoting } x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y \text{ and } s=x+y+z\right)$$

$$\Leftrightarrow 176x^4(y+z)-160x^3(y+z)^2+1024x^3yz-24\left((y+z)^3-3yz(y+z)\right)-456x^2yz(y+z)+456x^2yz(y+z)+56x((y+z)^2-2yz)^2-160xyz(y+z)^2-224xy^2z^2+419(y+z)(((y+z)^2-2yz)^2+y^2z^2-yz(y+z)^2)-1201yz\left((y+z)^3-3yz(y+z)\right)+1214y^2z^2(y+z)\overset{?}{\ge}0 \Leftrightarrow 176x^4.mx-160x^3.m^2x^2+1024x^3.nx^2-24x^2(m^3x^3-3nx^2.mx)-456x^2.nx^2.mx+56x(m^2x^2-2nx^2)^2-160x.nx^2.m^2x^2-224x.n^2x^4+419mx((m^2x^2-2nx^2)^2+n^2x^4-nx^2.m^2x^2)-1201.nx^2.(m^3x^3-3nx^2.mx)+1214.n^2x^4.mx\overset{?}{\ge}0 \left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right)$$

$$\Leftrightarrow 6912mn^2-(3296m^3+384m^2+384m-1024)n+419m^5+56m^4-24m^3-160m^2+176m \overset{?}{\underset{(*)}{\ge}} 0$$

Now, since $56m^4-24m^3-160m^2+176m=$

$$\frac{m}{4}\left(\left(419m^2+475m+\frac{1385}{4}\right)(2m-1)^2+270m+\frac{1431}{4}\right)>0$$

$\therefore$ $(*)$ is trivially true when : $3296m^3+384m^2+384m-1024\le 0$ and so we now focus on the case when : $3296m^3+384m^2+384m-1024>0$ and now, let us assume : $11m-4\le 0$ and then : $3296m^3+384m^2+384m-1024$

$$=\frac{32}{121}\left(\left(1133m^2+544m+\frac{3628}{11}\right)(11m-4)-\frac{28080}{11}\right)<0, \text{but :}$$

$3296m^3+384m^2+384m-1024>0$ and so, our assumption is incorrect and hence : $11m-4>0 \rightarrow (1)$

Now, LHS of $(*)$ is a quadratic expression in "n" with discriminant $\delta=$

$$(3296m^3+384m^2+384m-1024)^2-$$
$$27648m(419m^5+56m^4-24m^3-160m^2+176m)$$
$$\stackrel{?}{=}(m+1)^3(m-2)^2(4-11m)\le 0\ \left(\because 4-11m<0\ via\ ①\right)\Rightarrow(*)\text{ is true}$$
$$and\text{ so}, n_a\le\frac{3\sqrt{3}.(5b^2-6bc+5c^2)}{16s}\ \forall\ \Delta\ ABC,$$
$$''=''\text{ iff } m=2\Rightarrow''=''\text{ iff } b+c=2a\ (QED)$$