ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$m_a \leq \frac{\sqrt{s^2-11r^2}}{4r}\cdot h_a$$

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$$m_a \overset{?}{\leq} \frac{\sqrt{s^2-11r^2}}{4r}.h_a \Leftrightarrow \frac{a(2m_a)}{4rs} \overset{?}{\leq} \frac{\sqrt{s^2-11r^2}}{4r}$$

$$\Leftrightarrow a^2(2b^2+2c^2-a^2) \overset{?}{\leq} s^2(s^2-11r^2)$$

$$\Leftrightarrow 16a^2(2b^2+2c^2-a^2) \overset{?}{\leq} (a+b+c)^4 - 11\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right)$$

$$\Leftrightarrow 7a^4+3b^4+3c^4+\sum_{cyc} a^3b + \sum_{cyc} ab^3 - 12a^2(b^2+c^2) - 4b^2c^2 +$$

$$3abc\sum_{cyc} a \overset{?}{\geq} 0 \Leftrightarrow 7(y+z)^4+3(z+x)^4+3(x+y)^4+\sum_{cyc}(y+z)^3(z+x) +$$

$$\sum_{cyc}(y+z)(z+x)^3 - 12(y+z)^2((z+x)^2+(x+y)^2) - 4(z+x)^2(x+y)^2 +$$

$$6(y+z)(z+x)(x+y)(x+y+z) \overset{?}{\geq} 0 \left(\begin{matrix}\text{substituting :}\\ a=y+z, b=z+x, c=x+y\end{matrix}\right)$$

$$\Leftrightarrow x^4+4x^3(y+z)+2x^2(y+z)^2-11x^2yz-11xyz(y+z)+4yz(y+z)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow x^4+4x^3.mx+2x^2.m^2x^2-11x^2.nx^2-11x.nx^2.mx+4nx^2.m^2x^2 \overset{?}{\geq} 0$$

$$\left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right) \Leftrightarrow (4n+2)m^2-(11n-4)m+1-11n \underset{(*)}{\overset{?}{\geq}} 0$$

Now, LHS of $(*)$ is a quadratic expression in "m" with discriminant $\delta =$
$(11n-4)^2 - 4(4n+2)(1-11n) \boxed{=} 297n^2-16n+8 > 0$
$(\because \delta_0 = 256n^2 - 32(297n^2) < 0)$ & so, in order to prove $(*)$, it suffices to prove :

$$2(4n+2)m \overset{?}{\geq} 11n-4+\sqrt{\delta} \text{ and } \because \frac{(y+z)^2}{x^2} \overset{AM-GM}{\geq} \frac{4yz}{x^2} \therefore m^2 \geq 4n$$

$$\Rightarrow m \geq 2\sqrt{n} \text{ and so, it suffices to prove : } 2(4n+2)(2\sqrt{n}) \boxed{\overset{?}{\geq}} 11n-4+\sqrt{\delta}$$

$$\Leftrightarrow 4t(4t^2+2)+4-11t^2 \overset{?}{\geq} \sqrt{297t^4-16t^2+8}\ (\sqrt{n}=t)$$

$$\Leftrightarrow 16t^3-11t^2+8t+4 \overset{?}{\geq} \sqrt{297t^4-16t^2+8}$$

$$\Leftrightarrow (16t^3-11t^2+8t+4)^2 \overset{?}{\geq} 297t^4-16t^2+8$$

$$\left(\begin{matrix}\because \text{discriminant of } 16t^2-11t+8 = 121-512 < 0 \Rightarrow 16t^3-11t^2+8t > 0\\ \Rightarrow 16t^3-11t^2+8t+4 > 0\end{matrix}\right)$$

$$\Leftrightarrow 8(2t+1)(8t+1)(2t^2+1)(t-1)^2 \boxed{\overset{?}{\geq}} 0 \rightarrow \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore m_a \leq \frac{\sqrt{s^2-11r^2}}{4r}.h_a\ \forall \Delta\ ABC, "=" \text{ iff } y=z \text{ and } yz=x^2$$

$$\Rightarrow "=" \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$