ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$s^2 \geq 27r^2 \cdot \frac{b^2 - bc + c^2}{bc}$$

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$$s^2 \overset{?}{\geq} 27r^2 . \frac{b^2 - bc + c^2}{bc} \Leftrightarrow bc. s^3 \overset{?}{\geq} 27(s-a)(s-b)(s-c)(b^2 - bc + c^2)$$

$$\Leftrightarrow (z+x)(x+y)(x+y+z)^3 \overset{?}{\geq} 27xyz((z+x)^2 - (z+x)(x+y) + (x+y)^2)$$

$$\left(\begin{array}{c} x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \\ \text{and } s = x+y+z \end{array}\right)$$

$$\Leftrightarrow x^5 + 4(y+z)x^4 + 6x^3(y+z)^2 - 26x^3yz + 4x^2\left((y+z)^3 - 3yz(y+z)\right) -$$

$$12x^2yz(y+z) + x((y+z)^2 - 2yz)^2 + 77xy^2z^2 - 20xyz(y+z)^2 + yz(y+z)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow x^5 + 4mx.x^4 + 6x^3.m^2x^2 - 26x^3.nx^2 + 4x^2(m^3x^3 - 3nx^2.mx) -$$

$$12x^2.nx^2.mx + x(m^2x^2 - 2nx^2)^2 + 77x.n^2x^4 - 20x.nx^2.m^2x^2 + nx^2.m^3x^3$$

$$\left(m = \frac{y+z}{x}, n = \frac{yz}{x^2}\right) \Leftrightarrow 81n^2 + (m^3 - 24m^2 - 24m - 26)n + (m+1)^4 \overset{?}{\underset{(*)}{\geq}} 0$$

and it's trivially true when : $m^3 - 24m^2 - 24m - 26 > 0$ and so,

we now focus on the case when : $m^3 - 24m^2 - 24m - 26 \leq 0$, that is,

when : $m \leq m_0 \,\middle|\, m_0^3 - 24m_0^2 - 24m_0 - 26 = 0$

($\because m^3 - 24m^2 - 24m - 26$ has only one zero : $m_0 \approx 25$)

Now, LHS of $(*)$ is a quadratic expression in "n" with discriminant $\delta =$

$(m^3 - 24m^2 - 24m - 26)^2 - 324(m+1)^4 \boxed{=}$

$(m-2)^2(m-2)(m^3 - 24m^2 - 24m - 26 - 18(m+1)^2)$ and $\because$

$m^3 - 24m^2 - 24m - 26 \leq 0 \therefore \delta \leq 0$ if $m \geq 2$ (and $m \leq m_0$)

and in that case, LHS of $(*) \geq 0 \Rightarrow (*)$ is true and if $m < 2$, then : $\delta > 0$

and then, in order to prove $(*)$, it suffices to prove :

$$162n \overset{?}{\leq} -(m^3 - 24m^2 - 24m - 26) - \sqrt{\delta} \text{ and } \because \frac{(y+z)^2}{x^2} \overset{AM-GM}{\geq} \frac{4yz}{x^2}$$

$\therefore m^2 \geq 4n$ and so, it suffices to prove :

$$162.\frac{m^2}{4} \overset{?}{\leq} -(m^3 - 24m^2 - 24m - 26) - \sqrt{\delta}$$

$$\Leftrightarrow 2\sqrt{(m-2)^2(m-2)(m^3 - 24m^2 - 24m - 26 - 18(m+1)^2)} \overset{?}{\leq}$$

$$(2-m)(2m^2 + 37m + 26)$$

$$\Leftrightarrow (m-2)^2(2m^2 + 37m + 26)^2 \overset{?}{\geq}$$

$$4(m-2)^2(m-2)(m^3 - 24m^2 - 24m - 26 - 18(m+1)^2) \; (\because 2 - m > 0)$$

$$\Leftrightarrow 81(4m+1)(m^2 - 4)^2 \overset{?}{\geq} 0 \to \text{true} \Rightarrow (*) \text{ is true} \therefore s^2 \geq 27r^2 . \frac{b^2 - bc + c^2}{bc}$$

$\forall$ Δ ABC, " = " iff $y = z$ and $y + z = 2x \Rightarrow$ " = " iff Δ ABC is equilateral (QED)