ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$w_a \le \frac{3\sqrt{3}(b^2+c^2+6bc)}{32s}$$

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$$\frac{3\sqrt{3}.(b^2+c^2+6bc)}{32s} = \frac{3\sqrt{3}.((b+c)^2+4bc)}{32s} \overset{AM-GM}{\ge} \frac{12\sqrt{3}.(b+c).\sqrt{bc}}{32s}$$

$$\overset{?}{\ge} \frac{2\sqrt{bc}}{b+c}.\sqrt{s(s-a)} \Leftrightarrow 27(b+c)^4 \overset{?}{\ge} 256s^3(s-a) \Leftrightarrow 27(2s-a)^4 \overset{?}{\ge} 256s^3(s-a)$$

$$\Leftrightarrow 176t^4-608t^3+648t^2-216t+27 \overset{?}{\ge} 0 \ \left(t=\frac{s}{a}\right)$$

$$\Leftrightarrow (2t-3)^2\big((44t+24)(t-1)+27\big) \overset{?}{\ge} 0 \to \text{true} \because t>1$$

$$\therefore \frac{3\sqrt{3}.(b^2+c^2+6bc)}{32s} \ge \frac{2\sqrt{bc}}{b+c}.\sqrt{s(s-a)} = w_a \Rightarrow w_a \le \frac{3\sqrt{3}.(b^2+c^2+6bc)}{32s}$$

$\forall \Delta$ ABC, $''='' $ iff $b=c$ and $b+c=2a \Rightarrow ''=''$ iff Δ ABC is equilateral (QED)