

In any  $\Delta ABC$  holds :

$$\frac{|b - c|}{a} \leq \min \left\{ \frac{\sqrt{s^2 - 27r^2}}{s}, \sqrt{\frac{2(2s^2 - 27Rr)}{27r^2}} \right\}$$

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Let  $s_0 = \text{semiperimeter}$  (for own convenience),  $s = \sin \frac{A}{2}$ ,  $c = \cos \frac{B - C}{2}$

Case 1  $c \geq 2s$  and now,  $\frac{|b - c|}{a} \stackrel{?}{\leq} \sqrt{\frac{2(2s_0^2 - 27Rr)}{27r^2}}$

$$\Leftrightarrow \frac{1 - c^2}{1 - s^2} \stackrel{?}{\leq} \frac{4}{27} \cdot \frac{(1 - s^2) \frac{(c+s)^2}{4}}{s^2 \frac{(c-s)^2}{4}} - \frac{2R}{4Rs \left(\frac{c-s}{2}\right)}$$

$$\Leftrightarrow (27c^4s^2 - 54c^3s^3 + 31c^2s^4 + 8cs^5 + 4s^6) - (35c^2s^2 - 65cs^3 + 62s^4) +$$

$$(4c^2 - 19cs + 31s^2) \stackrel{?}{\geq} 0 \Leftrightarrow s^6(27t^4 - 54t^3 + 31t^2 + 8t + 4) -$$

$$s^4(35t^2 - 65t + 62) + s^2(4t^2 - 19t + 31) \stackrel{?}{\geq} 0 \quad \left(t = \frac{c}{s}\right)$$

$$\Leftrightarrow s^4(27t^2(t - 1)^2 + 4(t + 1)^2) - s^2(35t^2 - 65t + 62) + 4t^2 - 19t + 31 \stackrel{?}{\geq} 0 \quad (**)$$

Now, LHS of (\*\*) is a quadratic polynomial in "s<sup>2</sup>" with discriminant  $\delta = (35t^2 - 65t + 62)^2 - 4(27t^4 - 54t^3 + 31t^2 + 8t + 4)(4t^2 - 19t + 31) = -27(t - 1)^2(t - 2)^2((t - 2)(16t + 20) + 9) \leq 0$  ( $\because t \geq 2$ )  $\Rightarrow$  (\*\*)  $\Rightarrow$  (\*) is true

Case 2(a)  $c \leq 2s$  and  $s \leq \frac{1}{2}$  and then, to prove (\*\*), it suffices to prove :

$$2s^2(27t^4 - 54t^3 + 31t^2 + 8t + 4) \stackrel{?}{\leq} 35t^2 - 65t + 62 - \sqrt{\delta} \quad \& \quad \because s^2 \leq \frac{1}{4}, \text{ it suffices}$$

to prove :  $2(35t^2 - 65t + 62) - (27t^4 - 54t^3 + 31t^2 + 8t + 4) \stackrel{?}{\geq} 2\sqrt{\delta}$

$$\Leftrightarrow 3(2 - t)(9t^3 - 13t + 20) \stackrel{?}{\geq} 2\sqrt{\delta} \text{ and now, } 9t^3 - 13t + 20 =$$

$$\frac{(45t + 63)(10t - 7)^2 + 115t + 6913}{500} > 0 \text{ and } c \leq 2s \Rightarrow 2 - t \geq 0 \text{ and so, } \textcircled{1} \Leftrightarrow$$

$$9(2 - t)^2(9t^3 - 13t + 20)^2 \stackrel{?}{\geq} -108(t - 1)^2(2 - t)^2(16t^2 - 12t - 31)$$

$$\begin{aligned} & (\delta \text{ evaluated earlier}) \Leftrightarrow (9t^3 - 13t + 20)^2 + 12(t - 1)^2(16t^2 - 12t - 31) \stackrel{?}{\geq} 0 \\ & \Leftrightarrow \frac{((t - 1)^2(1296t^2 + 3888t + 6780) + 6960t - 5135)(2t - 1)^2 + 14412t + 147}{64} \stackrel{?}{\geq} 0 \end{aligned}$$

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$$\rightarrow \text{true} \because t = \frac{c}{s} = \frac{\frac{b+c}{a} \sin \frac{A}{2}}{\sin \frac{A}{2}} > 1 \Rightarrow \textcircled{1} \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

**Case 2(b)**  $c \leq 2s \wedge s \geq \frac{1}{2}$ ; Let :  $c - s = u \Rightarrow c = s + u$  & then :  $(*) \Leftrightarrow 27s^2u^4 + 54s^3u^3 + (31s^4 - 35s^2 + 4)u^2 + (16s^5 - 5s^3 - 11s)u + 16s^6 - 32s^4 + 16s^2 \stackrel{?}{\geq} 0$

$\because 27s^2(u - 1 + s)^4 \geq 0 \therefore$  to prove  $(\bullet)$ , it suffices to prove : LHS of  $(\bullet) \stackrel{?}{\geq} 27s^2(u - 1 + s)^4 \Leftrightarrow 54s^2(2 - s)u^3 - (131s^4 - 324s^3 + 197s^2 - 4)u^2 - (92s^5 - 324s^4 + 329s^3 - 108s^2 + 11s)u + 108s^5 - 11s^6 - 194s^4 + 108s^3 - 11s^2 \stackrel{?}{\geq} 0$ ;  $\because 54s^2(2 - s)u(u - 1 + s)^2 \geq 0 \therefore$  to prove  $(\bullet\bullet)$ , suffices to prove :

LHS of  $(\bullet\bullet) \stackrel{?}{\geq} 54s^2(2 - s)u(u - 1 + s)^2 \stackrel{\text{factoring out } (1-s)}{\Leftrightarrow} (1 + s)(23s^2 + 4)u^2 + (38s^4 - 70s^3 - 11s^2 - 11s)u + 11s^5 - 97s^4 + 97s^3 - 11s^2 \stackrel{?}{\geq} 0$

Discriminant  $\delta_0$  of LHS of  $(\bullet\bullet\bullet) = (38s^4 - 70s^3 - 11s^2 - 11s)^2 - 4(1 + s)(23s^2 + 4)(11s^5 - 97s^4 + 97s^3 - 11s^2)$

$\equiv 27(4s^4 + 28s^3 + 63s^2 + 2s + 11)(2s - 1)^2 \geq 0$  &  $\because u > 0 \therefore$  to prove  $(\bullet\bullet\bullet)$ , it suffices to prove :  $2(1 + s)(23s^2 + 4)u \stackrel{?}{\leq} - (38s^4 - 70s^3 - 11s^2 - 11s) - \sqrt{\delta_0}$  &  $\because u = c - s \leq 1 - s \therefore$  suffices to prove :  $\sqrt{\delta_0} \stackrel{?}{\leq} - (38s^4 - 70s^3 - 11s^2 - 11s) - 2(1 + s)(23s^2 + 4)(1 - s) = (2s - 1)(4s^3 + 37s^2 + 5s + 8) \Leftrightarrow 27(4s^4 + 28s^3 + 63s^2 + 2s + 11)(2s - 1)^2 \stackrel{?}{\leq} (2s - 1)^2(4s^3 + 37s^2 + 5s + 8)^2$  ( $\because s \geq \frac{1}{2}$ )  $\Leftrightarrow 4(1 - s)(s + 4)(23s^2 + 4)(s + 1)^2(2s - 1)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true} \Rightarrow (\bullet\bullet\bullet)\text{true}$

$\therefore$  combining all cases,  $\frac{|b - c|}{a} \leq \sqrt{\frac{2(2s_0^2 - 27Rr)}{27r^2}}$  & finally,  $\frac{|b - c|}{a} \stackrel{?}{\leq} \frac{\sqrt{s_0^2 - 27r^2}}{s_0}$

$\Leftrightarrow \frac{27s^2(c - s)^2}{(1 - s^2)(c + s)^2} \stackrel{?}{\leq} 1 - \frac{1 - c^2}{1 - s^2} \stackrel{\text{factoring out } (c-s)}{\Leftrightarrow} (c + s)^3 \stackrel{?}{\geq} 27s^2(c - s)$

$\Leftrightarrow (c + 7s)(c - 2s)^2 \stackrel{?}{\geq} 0 \rightarrow \text{true}$ ; so,  $\frac{|b - c|}{a} \leq \min \left\{ \frac{\sqrt{s^2 - 27r^2}}{s}, \sqrt{\frac{2(2s^2 - 27Rr)}{27r^2}} \right\}$

$\forall \Delta ABC$ , where  $s \equiv$  semiperimeter, " $=$ " iff  $\cos \frac{B - C}{2} = 2 \sin \frac{A}{2}$  (QED)