ROMANIAN MATHEMATICAL MAGAZINE

In any $\Delta$ ABC the following relationship holds :

$$|b-c| \le \min\left\{s.\sqrt{\frac{R-2r}{R}}, \sqrt{\frac{128}{135}(2s^2-27Rr)}\right\}$$

*Proposed by Dang Ngoc Minh-Vietnam*

*Solution by Soumava Chakraborty-Kolkata-India*

Let $s_0 =$ semiperimeter (for own convenience), $s = \sin\frac{A}{2}, c = \cos\frac{B-C}{2}$

and we first prove : $|b-c| \overset{?}{\le} s_0.\sqrt{\frac{R-2r}{R}} \Leftrightarrow \frac{16R^2.s^2(1-c^2)}{16R^2.(1-s^2)\frac{(c+s)^2}{4}} \overset{?}{\le} 1-4sc+4s^2$

$\Leftrightarrow 4s^3(c^3-s^3)+4cs^4(c-s)-(4c^3s-4s^4)-s^2(s^2+c^2-2cs)+$

$(c^2+2cs-3s^2) \overset{?}{\ge} 0 \Leftrightarrow 4s^3(c^2+s^2+cs)+4cs^4-4s(c^2+s^2+cs)-$

$s^2(c-s)+c+3s \overset{?}{\ge} 0 \left(\because c > s \text{ as } \cos\frac{B-C}{2} = \frac{b+c}{a}.\sin\frac{A}{2} > \sin\frac{A}{2}\right)$

$\Leftrightarrow (4s-4s^3)c^2-(8s^4-5s^2+1)c+3s^3-3s-4s^5 \overset{?}{\underset{(*)}{\le}} 0$ and $\because$ discriminant

$\delta = (8s^4-5s^2+1)^2-4(4s-4s^3)(3s^3-3s-4s^5)$

$= (32s^2(s^2-1)^2+(3s^2+1)^2) > 0$ and $4s-4s^3 > 0 \left(\text{as } 0 < \sin\frac{A}{2} < 1\right)$

$\therefore$ in order to prove $(*)$, it suffices to prove :

$2(4s-4s^3)c \overset{?}{\underset{(\bullet)}{\le}} 8s^4-5s^2+1+\sqrt{\delta}$ AND $2(4s-4s^3)c \overset{?}{\underset{(\bullet\bullet)}{\ge}} 8s^4-5s^2+1-\sqrt{\delta}$

Now, $2(4s-4s^3)c-(8s^4-5s^2+1) \overset{0<c\le 1}{\le} 2(4s-4s^3)-(8s^4-5s^2+1)$

and if it's $\le 0$, it's trivially $< \sqrt{\delta}$ and when it's $> 0$, then in order to prove $(\bullet)$,

it suffices to prove : $\delta \overset{?}{\ge} \left(2(4s-4s^3)-(8s^4-5s^2+1)\right)^2$

$\Leftrightarrow 16(1-s)(s+1)^4(2s-1)^2 \overset{?}{\ge} 0 \to$ true $\Rightarrow (\bullet)$ is true

Again, since $2(4s-4s^3)c > 0$ and $8s^4-5s^2+1 > 0 \therefore$ in order to prove $(\bullet\bullet)$,

it suffices to prove : $\delta \overset{?}{\ge} (8s^4-5s^2+1)^2$

$\Leftrightarrow 16s^2(1-s^2)((2s^2-1)^2+s^2+2) \overset{?}{\ge} 0 \to$ true (strict inequality) and so,

$(\bullet)$ and $(\bullet\bullet)$ are both true $\Rightarrow (*)$ is true $\therefore |b-c| \le s_0.\sqrt{\frac{R-2r}{R}}$

Again, $|b-c| \overset{?}{\le} \sqrt{\frac{128}{135}(2s_0^2-27Rr)}$

$$\Leftrightarrow \frac{16R^2.s^2(1-c^2)}{16R^2.(1-s^2)\frac{(c+s)^2}{4}} \overset{?}{\leq} \frac{256}{135} - \frac{128}{5}.\frac{4R^2.s.\frac{c-s}{2}}{16R^2.(1-s^2)\frac{(c+s)^2}{4}}$$

$$\Leftrightarrow (71s^2+64)c^2 - (128s^3+304s)c + 361s^2 - 64s^4 \boxed{\underset{(**)}{\overset{?}{\geq}}} 0 \text{ and } \because$$

$$\text{discriminant } \delta_0 = (128s^3+304s)^2 - 4(71s^2+64)(361s^2-64s^4)$$

$$= 108s^4(320s^2-77) \text{ which is clearly } \leq 0 \text{ for } s \in \left(0, \sqrt{\frac{77}{320}}\right]$$

$$\therefore \text{ we now focus on } s \in \left(\sqrt{\frac{77}{320}}, 1\right) \text{ and then, in order to prove } (**),$$

$$\text{it suffices to prove : } 2(71s^2+64)c \overset{?}{\leq} 128s^3 + 304s - \sqrt{108s^4(320s^2-77)} \text{ and}$$
$$\because c \leq 1 \therefore \text{ it suffices to prove :}$$
$$\sqrt{108s^4(320s^2-77)} \overset{?}{\leq} 128s^3 + 304s - 2(71s^2+64)$$
$$\Leftrightarrow 4(64s^3 - 71s^2 + 152s - 64)^2 \overset{?}{\geq} 108s^4(320s^2-77)$$
$$\left(\because 64s^3 - 71s^2 + 152s - 64 > 0 \ \forall s \in \left(\sqrt{\frac{77}{320}}, 1\right)\right)$$

$$\Leftrightarrow 64(1-s)(s+4)(2s-1)^2(71s^2+64) \overset{?}{\geq} 0 \to \text{true} \because s \in \left(\sqrt{\frac{77}{320}}, 1\right)$$

$$\Rightarrow (**) \text{ is true} \therefore |b-c| \leq \sqrt{\frac{128}{135}(2s_0^2 - 27Rr)} \text{ and so, whenever } "s" \text{ is}$$

$$\text{the semiperimeter, } |b-c| \leq s.\sqrt{\frac{R-2r}{R}}, \sqrt{\frac{128}{135}(2s^2-27Rr)} \ \therefore |b-c| \leq$$

$$\min\left\{s.\sqrt{\frac{R-2r}{R}}, \sqrt{\frac{128}{135}(2s^2-27Rr)}\right\} \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$