

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$(m_a + m_b)(m_b + m_c)(m_c + m_a) \leq \sqrt{\frac{27}{32} \cdot \left(3 \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} b^2 c^2 \right) + 5a^2 b^2 c^2 \right)}$$

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$$\begin{aligned} & \frac{3}{32} \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} b^2 c^2 \right) + \frac{10}{27} \cdot m_a^2 m_b^2 m_c^2 \stackrel{?}{\geq} \frac{1}{64} \cdot (a+b)^2 (b+c)^2 (c+a)^2 \\ & \Leftrightarrow \frac{3}{32} (2(s^2 - 4Rr - r^2)) ((s^2 + 4Rr + r^2)^2 - 16Rrs^2) + \\ & \frac{10}{27} s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \\ & \frac{16}{27} \stackrel{?}{\geq} \frac{1}{64} \cdot 4s^2(s^2 + 4Rr + r^2)^2 \end{aligned}$$

(Reference : Solution to Inequality in Triangle by Dang Ngoc Minh – 124;
published at www.ssmrmh.ro)

$$\Leftrightarrow 64s^6 - (1200Rr - 357r^2)s^4 + r^2(3180R^2 - 660Rr - 438r^2)s^2 - 91r^3(4R + r)^3 \stackrel{?}{\geq} 0 \text{ and } \therefore 64(s^2 - 16Rr + 5r^2)^3 \stackrel{\text{Gerretsen}}{\geq} 0$$

$$\begin{aligned} \therefore \text{in order to prove } (*), \text{ it suffices to prove : LHS of } (*) & \stackrel{?}{\geq} 64(s^2 - 16Rr + 5r^2)^3 \\ & \Leftrightarrow (208R - 67r)s^4 - r(5108R^2 - 3340Rr + 582r^2)s^2 + \\ & r^2(28480R^3 - 27792R^2r + 8412Rr^2 - 899r^3) \stackrel{?}{\geq} 0 \text{ and } \therefore \end{aligned}$$

$$\begin{aligned} (208R - 67r)(s^2 - 16Rr + 5r^2)^2 & \stackrel{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (**), \text{ it suffices} \\ \text{to prove : LHS of } (**)& \stackrel{?}{\geq} (208R - 67r)(s^2 - 16Rr + 5r^2)^2 \\ \Leftrightarrow (387R^2 - 221Rr + 22r^2)s^2 & \stackrel{?}{\geq} r(6192R^3 - 5660R^2r^2 + 1877Rr^2 - 194r^3) \end{aligned}$$

$$\begin{aligned} \text{Finally, LHS of } (***) & \stackrel{\text{Gerretsen}}{\geq} (387R^2 - 221Rr + 22r^2)(16Rr - 5r^2) \stackrel{?}{\geq} \\ \text{RHS of } (***) & \Leftrightarrow 21(R - 2r)(9R - 2r) \stackrel{?}{\geq} 0 \rightarrow \text{true } \therefore R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (***) \Rightarrow (***) \Rightarrow \end{aligned}$$

$$(*) \text{ is true } \therefore \sqrt{\frac{3}{32} \cdot \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} b^2 c^2 \right) + \frac{10}{27} \cdot m_a^2 m_b^2 m_c^2} \geq \frac{1}{8} (a+b)(b+c)(c+a)$$

and implementing it on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose medians as a consequence of trivial calculations $= \frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ respectively, we get :

$$\begin{aligned}
 & \sqrt{\frac{3}{32} \cdot \left(\frac{4}{9} \cdot \sum_{\text{cyc}} m_a^2 \right) \left(\frac{16}{81} \cdot \sum_{\text{cyc}} m_b^2 m_c^2 \right) + \frac{10}{27} \cdot \frac{1}{64} \cdot a^2 b^2 c^2} \geq \frac{1}{8} \cdot \frac{8}{27} \prod_{\text{cyc}} (m_b + m_c) \\
 \Rightarrow & \sqrt{\frac{3}{32} \cdot \left(\frac{4}{9} \cdot \frac{3}{4} \cdot \sum_{\text{cyc}} a^2 \right) \left(\frac{16}{81} \cdot \frac{9}{16} \cdot \sum_{\text{cyc}} b^2 c^2 \right) + \frac{10}{27} \cdot \frac{1}{64} \cdot a^2 b^2 c^2} \geq \frac{1}{27} \prod_{\text{cyc}} (m_b + m_c) \\
 \Rightarrow & \sqrt{\frac{81}{32} \cdot \left(\sum_{\text{cyc}} a^2 \right) \left(\sum_{\text{cyc}} b^2 c^2 \right) + \frac{135}{32} \cdot a^2 b^2 c^2} \geq (m_a + m_b)(m_b + m_c)(m_c + m_a) \\
 \therefore (m_a + m_b)(m_b + m_c)(m_c + m_a) & \leq \sqrt{\frac{27}{32} \cdot \left(3 \left(\sum_{\text{cyc}} a^2 \right) \left(\sum_{\text{cyc}} b^2 c^2 \right) + 5a^2 b^2 c^2 \right)} \\
 & \forall ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}
 \end{aligned}$$