ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{m_a m_b}+\sqrt{m_b m_c}+\sqrt{m_c m_a}\le\sqrt{\frac{74(a^2b^2+b^2c^2+c^2a^2)+7(a^4+b^4+c^4)}{12(a^2+b^2+c^2)}}$$

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$$\sum_{cyc}\sqrt{bc}=\sum_{cyc}\sqrt{\frac{bc}{b+c}.(b+c)}\overset{CBS}{\le}\sqrt{\sum_{cyc}\frac{bc}{b+c}}.\sqrt{\sum_{cyc}(b+c)}$$

$$=\sqrt{\frac{4s}{2s(s^2+2Rr+r^2)}.\sum_{cyc}\left(bc\left(a^2+\sum_{cyc}ab\right)\right)}=\sqrt{\frac{2(8Rrs^2+(s^2+4Rr+r^2)^2)}{s^2+2Rr+r^2}}$$

$$\overset{?}{\le}2.\sqrt{\frac{7\left(\sum_{cyc}a^2\right)^2+60\sum_{cyc}a^2b^2}{36\sum_{cyc}a^2}}$$

$$=\sqrt{\frac{7(s^2-4Rr-r^2)^2+15((s^2+4Rr+r^2)^2-16Rrs^2)}{18(s^2-4Rr-r^2)}}$$

$$\Leftrightarrow 13s^6-(240Rr-29r^2)s^4+r^2(432R^2+176Rr+47r^2)s^2+$$
$$r^3(1280R^3+1136R^2r^2+328Rr^2+31r^3)\underset{(*)}{\overset{?}{\ge}}0 \text{ and } \because 13(s^2-16Rr+5r^2)^3$$

$$\overset{Gerretsen}{\ge}0\ \therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*)\overset{?}{\ge}$$
$$13(s^2-16Rr+5r^2)^3\Leftrightarrow(192R-83r)s^4-r(4776R^2-3208Rr+464r^2)s^2+$$
$$r^2(27264R^3-24392R^2r^2+7964Rr^2-797r^3)\underset{(**)}{\overset{?}{\ge}}0 \text{ and } \because$$

$$(192R-83r)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\ge}0\ \therefore \text{ in order to prove } (**),$$
$$\text{it suffices to prove : LHS of } (**)\overset{?}{\ge}(192R-83r)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow(228R^2-228Rr+61r^2)s^2\underset{(***)}{\overset{?}{\ge}}r(3648R^3-4596R^2r^2+1686Rr^2-213r^3)$$

$$\text{Now, }(R-r)(s^2-16Rr+5r^2)\overset{Rouche}{\ge}(R-r)\left(\begin{matrix}2R^2-6Rr+4r^2-\\2(R-2r)\sqrt{R^2-2Rr}\end{matrix}\right)$$
$$=(R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2+r^2\right)\ge r^2(R-2r)\ \left(\because R-2r\overset{Euler}{\ge}0\right)$$

$$\Rightarrow s^2 \geq 16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r} \therefore \text{LHS of } (***) \geq$$

$$(228R^2 - 228Rr + 61r^2)\left(16Rr - 5r^2 + \frac{r^2(R-2r)}{R-r}\right) \overset{?}{\geq} \text{RHS of } (***)$$

$$\Leftrightarrow 36t^3 - 62t^2 - 5t - 30 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)(36t^2 + 10t + 15) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{\text{cyc}} \sqrt{bc} \leq 2.\sqrt{\frac{7\left(\sum_{\text{cyc}} a^2\right)^2 + 60\sum_{\text{cyc}} a^2b^2}{36\sum_{\text{cyc}} a^2}} \text{ and implementing it on}$$

$$a \text{ triangle with sides } \frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}, \text{we arrive at :}$$

$$\frac{2}{3}.\sum_{\text{cyc}} \sqrt{m_b m_c} \leq 2.\sqrt{\frac{7.\frac{16}{81}.\left(\sum_{\text{cyc}} m_a^2\right)^2 + 60.\frac{16}{81}.\sum_{\text{cyc}} m_a^2 m_b^2}{36.\frac{4}{9}.\sum_{\text{cyc}} m_a^2}}$$

$$= 2.\sqrt{\frac{7.\frac{16}{81}.\left(\frac{3}{4}.\sum_{\text{cyc}} a^2\right)^2 + 60.\frac{16}{81}.\frac{9}{16}.\sum_{\text{cyc}} a^2b^2}{36.\frac{4}{9}.\frac{3}{4}.\sum_{\text{cyc}} a^2}} = 2.\sqrt{\frac{74\sum_{\text{cyc}} a^2b^2 + 7\sum_{\text{cyc}} a^4}{108\sum_{\text{cyc}} a^2}}$$

$$\Rightarrow \sum_{\text{cyc}} \sqrt{m_b m_c} \leq \sqrt{\frac{74\sum_{\text{cyc}} a^2b^2 + 7\sum_{\text{cyc}} a^4}{\frac{108}{9}.\sum_{\text{cyc}} a^2}} \text{ and so,}$$

$$\sqrt{m_a m_b} + \sqrt{m_b m_c} + \sqrt{m_c m_a} \leq \sqrt{\frac{74(a^2b^2 + b^2c^2 + c^2a^2) + 7(a^4 + b^4 + c^4)}{12(a^2 + b^2 + c^2)}}$$

$$\forall \text{ ABC}, '' = '' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$