ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sqrt{h_a h_b}+\sqrt{h_b h_c}+\sqrt{h_c h_a}\le \frac{8}{9}.R+\frac{65}{9}.r$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{h_b h_c}=\frac{1}{2R}.\sum_{cyc}\sqrt{ca.ab}$$

$$=\frac{1}{2R}.\sum_{cyc}\left(\frac{a}{\sqrt{(s-b)(s-c)}}.\sqrt{bc(s-b)(s-c)}\right)$$

$$\overset{CBS}{\le}\frac{1}{2R}.\sqrt{\sum_{cyc}\frac{a^2(s-a)}{r^2 s}}.\sqrt{r^2 s^2.\sum_{cyc}\frac{bc}{s(s-a)}}$$

$$=\frac{1}{2R}.\sqrt{\frac{2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{s}}.\sqrt{s^2.\sum_{cyc}\sec^2\frac{A}{2}}$$

$$=\frac{1}{2R}.\sqrt{4Rr+4r^2}.\sqrt{s^2.\frac{(4R+r)^2+s^2}{s^2}}\overset{?}{\le}\frac{8R+65r}{9}$$

$$\Leftrightarrow (16R^2+130Rr)^2\ \underset{(*)}{\overset{?}{\ge}}\ 81(4Rr+4r^2)((4R+r)^2+s^2)$$

$$\text{Now, LHS of }(*)\overset{Blundon-Gerretsen}{\le}81(4Rr+4r^2)\left((4R+r)^2+\frac{R(4R+r)^2}{4R-2r}\right)$$

$$\overset{?}{\le}(16R^2+130Rr)^2$$

$$\Leftrightarrow 256t^5-2448t^4+7692t^3-8207t^2+1053t+162\ \underset{(**)}{\overset{?}{\ge}}\ 0\ \left(t=\frac{R}{r}\right)$$

Case 1 $t\ge\frac{16689}{4309}$ and then : LHS of $(**)=$

$$(t-2)\left(\left(\frac{64t^2-4t+25}{4}\right)(4t-15)^2+\frac{1632t-5949}{4}\right)>0$$

$$\left(\because t\ge\frac{16689}{4309}>\frac{5949}{1632}>2\right)\Rightarrow(**)\text{ is true}$$

Case 2 $t<\frac{16689}{4309}$ and then : LHS of $(**)=$

$$(t-2)\left(\left(\frac{641t^2+88t(t-2)+39(t^2-4)}{27}\right)(3t-11)^2+\frac{16689-4309t}{27}\right)\ge 0$$

$$\left(\because 2\overset{Euler}{\le}t<\frac{16689}{4309}\right)\Rightarrow(**)\text{ is true}\therefore\text{combining both cases, }(**)\Rightarrow(*)$$

is true $\forall\ \Delta\ ABC$ ***and*** **so,** $\sqrt{h_a h_b}+\sqrt{h_b h_c}+\sqrt{h_c h_a}\le \frac{8}{9}.R+\frac{65}{9}.r\ \forall\ ABC,$

$''=''$ **iff** Δ **ABC is equilateral (QED)**