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In any Δ ABC the following relationship holds :

$$\frac{s-n_a-n_b-n_c}{2r}+\sum_{cyc}\frac{n_a}{h_a}\ge\sum_{cyc}\left(\left(\frac{n_a-\sqrt{4r^2+(b-c)^2}}{\sqrt{4r^2+(b-c)^2}}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\left(\frac{n_a-\sqrt{4r^2+(b-c)^2}}{\sqrt{4r^2+(b-c)^2}}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\right)\overset{\text{Bogdan Fustei}}{=}$$

$$\sum_{cyc}\left(\left(\frac{n_a-\frac{an_a}{s}}{\frac{an_a}{s}}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\right)=\sum_{cyc}\left(\frac{s-a}{a}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)$$

$$=\sum_{cyc}\left(\frac{s-a}{a}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}\right)-\sum_{cyc}\frac{sn_a}{2rs}+\sum_{cyc}\frac{n_a}{h_a}\overset{?}{\le}\frac{s-n_a-n_b-n_c}{2r}+\sum_{cyc}\frac{n_a}{h_a}$$

$$\Leftrightarrow\frac{s}{2r}\underset{(*)}{\overset{?}{\ge}}\sum_{cyc}\left(\left(\frac{s-a}{a}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}\right)\right)$$

$$\text{Now, }\sum_{cyc}\left(\left(\frac{s-a}{a}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}\right)\right)=\frac{1}{abc}\sum_{cyc}\left((s-a)(bm_c+cm_b)\right)$$

$$=\frac{2rs}{4Rrs}.\sum_{cyc}\left((s-a)\left(\frac{m_c}{h_b}+\frac{m_b}{h_c}\right)\right)\le\frac{1}{2R}.\sum_{cyc}\left((s-a)\left(\frac{R}{r}\right)\right)$$

$$\left(\begin{array}{c}\because\frac{R}{r}\ge\frac{m_b}{h_c}+\frac{m_c}{h_b}\text{ and analogs}\to\text{reference : article titled}\\ \text{"New Triangle Inequalities With Brocard's Angle"}\\ \text{by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 – 7,}\\ \text{published at : www.ssmrmh.ro}\end{array}\right)$$

$$=\frac{1}{2r}\sum_{cyc}(s-a)=\frac{s}{2r}\Rightarrow(*)\text{ is true}\therefore\frac{s-n_a-n_b-n_c}{2r}+\sum_{cyc}\frac{n_a}{h_a}$$

$$\ge\sum_{cyc}\left(\left(\frac{n_a-\sqrt{4r^2+(b-c)^2}}{\sqrt{4r^2+(b-c)^2}}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\right)\ \forall\,\Delta\text{ ABC},$$

$$\text{"="}\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$