ROMANIAN MATHEMATICAL MAGAZINE

For $x = (4r^2 + (b-c)^2)^{\frac{1}{2}}$ and analogs, in any Δ ABC, the following relationship holds :

$$\frac{AI + BI + CI}{r} \le \sum_{cyc} \sqrt{\frac{yz}{\left(p_b\left(\frac{l_b}{g_b}\right)^{\frac{1}{2}} - y\right)\left(p_c\left(\frac{l_c}{g_c}\right)^{\frac{1}{2}} - z\right)}}$$

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$$m_a n_a \overset{?}{\ge} p_a^2 + \frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\\text{and}\\\text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\ge} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$
$$+ \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9}.\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\ge} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$
$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a).\frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9}\right) +$$
$$\left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\ge} 0 \ (\because (b-c)^2 \ge 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} +$$
$$\frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3\right)}{36a(2s+a)^2} +$$
$$\frac{(s-a)\left((s-a)(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3) + 1701a^4\right)}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\ge} 0$$

$$\to \text{true (strict inequality)} \therefore m_a n_a \ge p_a^2 + \frac{(b-c)^2}{18} \ge p_a^2 \Rightarrow m_a n_a \ge p_a^2 \to (1)$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\ge} m_a l_a \to (2) \therefore (1)\bullet(2) \Rightarrow (m_a n_a)(n_a g_a) \ge p_a^2.m_a l_a$$

$$\Rightarrow n_a \ge p_a \cdot \sqrt{\frac{l_a}{g_a}} \text{ and analogs and also, } x = (4r^2 + (b-c)^2)^{\frac{1}{2}}$$

$$\overset{\text{Bogdan Fustei}}{=} \frac{an_a}{s} \text{ and analogs} \therefore \sum_{cyc} \sqrt{\frac{yz}{\left(p_b\left(\frac{l_b}{g_b}\right)^{\frac{1}{2}} - y\right)\left(p_c\left(\frac{l_c}{g_c}\right)^{\frac{1}{2}} - z\right)}} =$$

$$\sum_{cyc} \frac{1}{\sqrt{\left(\frac{p_b\left(\frac{l_b}{g_b}\right)^{\frac{1}{2}}}{y} - 1\right)\left(\frac{p_c\left(\frac{l_c}{g_c}\right)^{\frac{1}{2}}}{z} - 1\right)}} \ge \sum_{cyc} \frac{1}{\sqrt{\left(\frac{n_b}{\left(\frac{bn_b}{s}\right)} - 1\right)\left(\frac{n_c}{\left(\frac{cn_c}{s}\right)} - 1\right)}} =$$

$$\sum_{cyc} \sqrt{\frac{bc}{(s-b)(s-c)}} = \sum_{cyc} \frac{1}{\sin\frac{A}{2}} = \frac{1}{r} \cdot \sum_{cyc} \frac{r}{\sin\frac{A}{2}} = \frac{1}{r} \cdot \sum_{cyc} AI \therefore \frac{AI + BI + CI}{r} \le$$

$$\sum_{cyc} \sqrt{\frac{yz}{\left(p_b\left(\frac{l_b}{g_b}\right)^{\frac{1}{2}} - y\right)\left(p_c\left(\frac{l_c}{g_c}\right)^{\frac{1}{2}} - z\right)}} \ \forall \Delta ABC, \text{"="} \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$