ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{g_a+g_b+g_c}{2r}+\frac{1}{2}\cdot\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\geq$$

$$\geq 2\sum_{cyc}\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum_{cyc}\frac{h_a}{n_a+s}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2\sum_{cyc}\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum_{cyc}\frac{h_a}{n_a+s}=\frac{g_a+g_b+g_c+s}{2r}$$

(Reference : Identity in Triangle by Bogdan Fustei – 4; published at www.ssmrmh.ro)

$\therefore$ the main inequality becomes : $\frac{g_a+g_b+g_c}{2r}+$

$$\frac{1}{2}\cdot\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\overset{?}{\geq}\frac{g_a+g_b+g_c+s}{2r}$$

$$\Leftrightarrow\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\overset{?}{\underset{(*)}{\geq}}\frac{s}{r}$$

$$\text{Now, }\left(\frac{s}{r}\right)^2\leq\sqrt{\frac{l_al_bl_c}{g_ag_bg_c}}\prod_{cyc}\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}$$

(Reference : Inequality in Triangle by Bogdan Fustei – 109; published at www.ssmrmh.ro)

$$\Rightarrow\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\geq\frac{s}{r}\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{g_a+g_b+g_c}{2r}+\frac{1}{2}\cdot\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\geq$$

$$2\sum_{cyc}\frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} + \sum_{cyc}\frac{h_a}{n_a + s}\ \forall\ \Delta\ ABC,$$

$'' ='' \text{ iff } \Delta\ ABC$ **is equilateral (QED)**