ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\left(\frac{s}{r}\right)^2 \le \left(\frac{l_a l_b l_c}{g_a g_b g_c}\right)^{\frac{1}{2}} \prod_{cyc} \frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}} - (4r^2 + (b-c)^2)^{\frac{1}{2}}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a n_a \overset{?}{\ge} p_a^2 + \frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\ge} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$
$$+ \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9}.\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\ge} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$
$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a).\frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9}\right) +$$
$$\left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\ge} 0 \; (\because (b-c)^2 \ge 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} +$$
$$\frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3\right)}{36a(2s+a)^2} +$$
$$\frac{(s-a)\left((s-a)(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3) + 1701a^4\right)}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\ge} 0$$

$$\rightarrow \text{true (strict inequality)} \therefore m_a n_a \ge p_a^2 + \frac{(b-c)^2}{18} \ge p_a^2 \Rightarrow m_a n_a \ge p_a^2 \rightarrow (1)$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\ge} m_a l_a \rightarrow (2) \therefore (1)\bullet(2) \Rightarrow (m_a n_a)(n_a g_a) \ge p_a^2.m_a l_a$$

$$\Rightarrow n_a \ge p_a.\sqrt{\frac{l_a}{g_a}} \Rightarrow \left(\frac{l_a}{g_a}\right)^{\frac{1}{2}}.\frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}} - (4r^2 + (b-c)^2)^{\frac{1}{2}}} = \frac{1}{1 - \frac{\sqrt{4r^2+(b-c)^2}}{p_a.\sqrt{\frac{l_a}{g_a}}}}$$

$$\overset{\text{Bogdan Fustei}}{=} \frac{1}{1-\frac{\left(\frac{an_a}{s}\right)}{p_a\cdot\sqrt{\frac{l_a}{g_a}}}} \geq \frac{1}{1-\frac{\left(\frac{an_a}{s}\right)}{n_a}} = \frac{s}{s-a} \therefore \left(\frac{l_a}{g_a}\right)^{\frac{1}{2}} \cdot \frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}} - \left(4r^2+(b-c)^2\right)^{\frac{1}{2}}} \geq$$

$$\frac{s}{s-a}\prod_{cyc}\frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}} - \left(4r^2+(b-c)^2\right)^{\frac{1}{2}}} \geq \prod_{cyc}\frac{s}{s-a} = \frac{s^3}{r^2 s} = \frac{s^2}{r^2} \text{ and so,}$$

$$\left(\frac{s}{r}\right)^2 \leq \left(\frac{l_a l_b l_c}{g_a g_b g_c}\right)^{\frac{1}{2}}\prod_{cyc}\frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}} - \left(4r^2+(b-c)^2\right)^{\frac{1}{2}}} \;\forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$