ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\frac{g_a}{r}+\frac{n_a}{r_a}=\frac{4(h_a-r)}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}$$

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$$g_a-n_a+\sqrt{4r^2+(b-c)^2}\overset{\text{Bogdan Fustei}}{=}g_a-n_a+\frac{an_a}{s}=g_a-\frac{r.n_a(s-a)}{rs}$$

$$=g_a-\frac{r.n_a}{r_a}=\frac{g_ar_a-r.n_a}{r_a}\ \therefore\left(\frac{g_a}{r}+\frac{n_a}{r_a}\right)\left(g_a-n_a+\sqrt{4r^2+(b-c)^2}\right)=$$

$$\left(\frac{g_ar_a+r.n_a}{r.r_a}\right)\left(\frac{g_ar_a-r.n_a}{r_a}\right)=\frac{g_a^2r_a^2-r^2n_a^2}{r.r_a^2}$$

$$\overset{\text{Bogdan Fustei}}{=}\frac{((s-a)^2+2r.h_a)r_a^2-r^2(s^2-2h_ar_a)}{r.r_a^2}$$

$$=\frac{(s-a)^2.\frac{r^2s^2}{(s-a)^2}+2r.h_ar_a^2-r^2s^2+2r^2.h_ar_a}{r.r_a^2}=\frac{2h_ar_a+2r.h_a}{r_a}=\frac{2h_a\left(\frac{rs}{s-a}+\frac{rs}{s}\right)}{r_a}$$

$$=\frac{2h_a(s-a)\frac{rs(2s-a)}{s(s-a)}}{rs}=2h_a\left(\frac{2s-a}{s}\right)=4h_a-\frac{4rs}{s}=4(h_a-r)$$

$$\therefore\frac{g_a}{r}+\frac{n_a}{r_a}=\frac{4(h_a-r)}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}\ \forall\,\Delta\text{ ABC (QED)}$$