ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{(n_a+s)\left(w_a+w_b+m_c-n_a.\sqrt{3}\right)}{n_a}\le 2r\sqrt{3}\sum_{cyc}\frac{h_a}{n_a-\sqrt{4r^2+(b-c)^2}}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{(n_a+s)\left(w_a+w_b+m_c-n_a.\sqrt{3}\right)}{n_a}\overset{\text{Lessel-Pelling}}{\le}$$

$$\le\sum_{cyc}\frac{(n_a+s)\left(s.\sqrt{3}-n_a.\sqrt{3}\right)}{n_a}=\sum_{cyc}\frac{\sqrt{3}.(s^2-n_a^2)}{n_a}\overset{\text{Bogdan Fustei}}{=}\sum_{cyc}\frac{\sqrt{3}.(2h_ar_a)}{n_a}$$

$$\therefore\sum_{cyc}\frac{(n_a+s)\left(w_a+w_b+m_c-n_a.\sqrt{3}\right)}{n_a}\le 2\sqrt{3}.\sum_{cyc}\frac{h_ar_a}{n_a}\to (1)\ \textit{and}$$

$$2r.\sqrt{3}.\sum_{cyc}\frac{h_a}{n_a-\sqrt{4r^2+(b-c)^2}}\overset{\text{Bogdan Fustei}}{=}2r.\sqrt{3}.\sum_{cyc}\frac{h_a}{n_a-\frac{an_a}{s}}=$$

$$=2\sqrt{3}.\sum_{cyc}\left(\left(\frac{rs}{s-a}\right)\left(\frac{h_a}{n_a}\right)\right)=$$

$$=2\sqrt{3}.\sum_{cyc}\frac{h_ar_a}{n_a}\overset{\text{via }(1)}{\ge}\sum_{cyc}\frac{(n_a+s)\left(w_a+w_b+m_c-n_a.\sqrt{3}\right)}{n_a}\ \textit{and so}$$

$$\sum_{cyc}\frac{(n_a+s)\left(w_a+w_b+m_c-n_a.\sqrt{3}\right)}{n_a}\le 2r.\sqrt{3}.\sum_{cyc}\frac{h_a}{n_a-\sqrt{4r^2+(b-c)^2}}\ \forall\,\Delta\,ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)