ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\left(s+p_a\sqrt{\frac{w_a}{g_a}}\right)\left(w_a+w_b+m_c-n_a\sqrt{3}\right)}{h_a}\le 2\sqrt{3}(4R+r)$$

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$$m_a n_a \overset{?}{\ge} p_a^2+\frac{(b-c)^2}{18}\overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\ge}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$
$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$
$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\ge}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$
$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$
$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$
$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\ge}0\ (\because (b-c)^2\ge 0)$$
$$\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$
$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\ge}0$$
$$\Leftrightarrow\frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$
$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)(b-c)^2}{324a(2s+a)^4}\overset{?}{\ge}0$$

$\rightarrow$ true (strict inequality) $\therefore m_a n_a\ge p_a^2+\frac{(b-c)^2}{18}\ge p_a^2\Rightarrow m_a n_a\ge p_a^2\rightarrow$ ① and

$n_a g_a\overset{\text{Bogdan Fustei}}{\ge} m_a w_a\rightarrow$ ② $\therefore$ ①•② $\Rightarrow (m_a n_a)(n_a g_a)\ge p_a^2.m_a w_a$

$$\Rightarrow \mathbf{n}_a \ge \sqrt{\frac{\mathbf{w}_a}{g_a}}.\mathbf{p}_a \Rightarrow \left(\mathbf{s}+\mathbf{p}_a.\sqrt{\frac{\mathbf{w}_a}{g_a}}\right)\left(\mathbf{w}_a+\mathbf{w}_b+\mathbf{m}_c-\mathbf{n}_a.\sqrt{3}\right) \overset{\text{Lessel-Pelling}}{\le}$$

$$\left(\mathbf{s}+\mathbf{p}_a.\sqrt{\frac{\mathbf{w}_a}{g_a}}\right)\left(\mathbf{s}.\sqrt{3}-\mathbf{n}_a.\sqrt{3}\right) \le (\mathbf{s}+\mathbf{n}_a)\left(\mathbf{s}.\sqrt{3}-\mathbf{n}_a.\sqrt{3}\right)$$

$$\left(\because 2\mathbf{h}_a\mathbf{r}_a \overset{\text{Bogdan Fustei}}{=} \mathbf{s}^2-\mathbf{n}_a^2 \Rightarrow \mathbf{s}-\mathbf{n}_a > 0\right)$$

$$\Rightarrow \frac{\left(\mathbf{s}+\mathbf{p}_a.\sqrt{\frac{\mathbf{w}_a}{g_a}}\right)\left(\mathbf{w}_a+\mathbf{w}_b+\mathbf{m}_c-\mathbf{n}_a.\sqrt{3}\right)}{\mathbf{h}_a} \le \frac{\sqrt{3}.\left(\mathbf{s}^2-\mathbf{n}_a^2\right)}{\mathbf{h}_a} \overset{\text{Bogdan Fustei}}{=} \frac{\sqrt{3}.2\mathbf{h}_a\mathbf{r}_a}{\mathbf{h}_a}$$

$$\therefore \frac{\left(\mathbf{s}+\mathbf{p}_a.\sqrt{\frac{\mathbf{w}_a}{g_a}}\right)\left(\mathbf{w}_a+\mathbf{w}_b+\mathbf{m}_c-\mathbf{n}_a.\sqrt{3}\right)}{\mathbf{h}_a} \le 2\sqrt{3}.\mathbf{r}_a \text{ and analogs}$$

$$\Rightarrow \sum_{\text{cyc}} \frac{\left(\mathbf{s}+\mathbf{p}_a.\sqrt{\frac{\mathbf{w}_a}{g_a}}\right)\left(\mathbf{w}_a+\mathbf{w}_b+\mathbf{m}_c-\mathbf{n}_a.\sqrt{3}\right)}{\mathbf{h}_a} \le 2\sqrt{3}.\sum_{\text{cyc}} \mathbf{r}_a = 2\sqrt{3}.(4\mathbf{R}+\mathbf{r})$$

$\forall\ \Delta\ ABC$, " = " iff $\Delta\ ABC$ is equilateral (QED)