ROMANIAN MATHEMATICAL MAGAZINE

In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\sin A+\sin B+\sin C=4\sin(\widehat{AIB})\sin(\widehat{BIC})\sin(\widehat{CIA})$$

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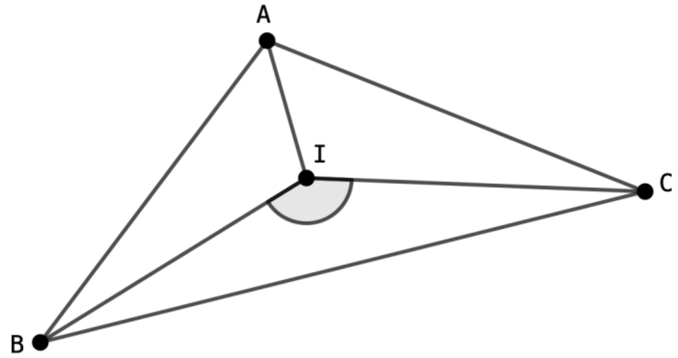

$$\sphericalangle BIC=180^\circ-\frac{\widehat{B}}{2}-\frac{\widehat{C}}{2}=180^\circ-\frac{\widehat{B}+\widehat{C}}{2}=$$

$$=180^\circ-\frac{180^\circ-\widehat{A}}{2}=180^\circ-90^\circ+\frac{\widehat{A}}{2}=90^\circ+\frac{\widehat{A}}{2}$$

$$\sin(\widehat{BIC})=\sin\left(90^\circ+\frac{A}{2}\right)=\sin 90^\circ\cos\frac{A}{2}+\sin\frac{A}{2}\cos 90^\circ=\cos\frac{A}{2}$$

Analogous: $\sin(\widehat{AIB})=\cos\frac{C}{2};\sin(\widehat{CIA})=\sin\frac{B}{2}$

$$\sin A+\sin B+\sin C=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\sin\left(2\cdot\frac{C}{2}\right)=$$

$$=2\sin\frac{180^\circ-C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}=$$

$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\cos\frac{C}{2}\cdot\cos\left(90^\circ-\frac{C}{2}\right)=$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{180^\circ-C}{2}\right)=$$

$$=2\cos\frac{C}{2}\cdot 2\cos\frac{A-B+180^\circ-C}{4}\cos\frac{A-B-180^\circ+C}{4}=$$

$$=4\cos\frac{C}{2}\cos\frac{A-B-C+A+B+C}{4}\cos\frac{A-B-A-B-C+C}{4}=$$

$$=4\cos\frac{C}{2}\cos\frac{2A}{4}\cos\left(-\frac{2B}{4}\right)=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=4\sin(\widehat{AIB})\sin(\widehat{BIC})\sin(\widehat{CIA})$$