ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\cot A+\cot B+\cot C=\frac{2(\sin^2 A+\sin^2 B+\sin^2 C)}{\sin 2A+\sin 2B+\sin 2C}$$

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$$\sin 2A+\sin 2B+\sin 2C=2\sin\frac{2A+2B}{2}\cos\frac{2A-2B}{2}+\sin 2C=$$

$$=2\sin(A+B)\cos(A-B)+2\sin C\cos C=$$

$$=2\sin(\pi-C)\cos(A-B)+2\sin C\cos C=2\sin C\left(\cos(A-B)+\cos C\right)=$$

$$=2\sin C\cdot 2\cos\frac{A-B+C}{2}\cos\frac{A-B-C}{2}=$$

$$=4\sin C\cos\frac{\pi-2B}{2}\cos\frac{2A-\pi}{2}=4\sin C\cos\left(\frac{\pi}{2}-B\right)\cos\left(\frac{\pi}{2}-A\right)=$$

$$=4\sin C\sin B\sin A$$

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C \quad (1)$$

$$\cot A+\cot B+\cot C=\sum_{cyc}\cot A=\sum_{cyc}\frac{\cos A}{\sin A}=\sum_{cyc}\frac{b^2+c^2-a^2}{2bc\sin A}=$$

$$=\frac{1}{4F}\sum_{cyc}(b^2+c^2-a^2)=\frac{1}{4F}\sum_{cyc}a^2=\frac{1}{4F}\sum_{cyc}4R^2\sin^2 A=$$

$$=\frac{4R^2(\sin^2 A+\sin^2 B+\sin^2 C)}{4\cdot 2R^2\sin A\sin B\sin C}=\frac{2(\sin^2 A+\sin^2 B+\sin^2 C)}{4\sin A\sin B\sin C}\overset{(1)}{=}$$

$$=\frac{2(\sin^2 A+\sin^2 B+\sin^2 C)}{\sin 2A+\sin 2B+\sin 2C}$$