

ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC the following relationship holds :

$$2 \sum_{\text{cyc}} \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b - c)^2}} + \sum_{\text{cyc}} \frac{h_a}{n_a + s} = \frac{g_a + g_b + g_c + s}{2r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} & \frac{2(h_a - r)}{g_a - n_a + \sqrt{4r^2 + (b - c)^2}} + \frac{h_a}{n_a + s} \stackrel{?}{=} \frac{g_a + s - a}{2r} \quad \text{Bogdan Fusteï} \Leftrightarrow \\ & \frac{2\left(\frac{2rs}{a} - r\right)}{g_a - n_a + \frac{an_a}{s}} + \frac{h_a(s - n_a)}{s^2 - n_a^2} \stackrel{?}{=} \frac{g_a + s - a}{2r} \quad \text{Bogdan Fusteï} \Leftrightarrow \\ & \frac{2rs\left(\frac{b+c}{a}\right)}{sg_a - (s - a)n_a} + \frac{h_a(s - n_a)}{2h_a r_a} \stackrel{?}{=} \frac{g_a + s - a}{2r} \\ & \Leftrightarrow \frac{2rs\left(\frac{b+c}{a}\right)}{sg_a - (s - a)n_a} \stackrel{?}{=} \frac{g_a + s - a}{2r} - \frac{(s - a)(s - n_a)}{2rs} \\ & \Leftrightarrow \frac{2rs\left(\frac{b+c}{a}\right)}{sg_a - (s - a)n_a} \stackrel{?}{=} \frac{sg_a + s(s - a) - s(s - a) + (s - a)n_a}{2rs} \\ & \Leftrightarrow 4r^2 s^2 \left(\frac{b+c}{a}\right) \stackrel{?}{=} (sg_a + (s - a)n_a)(sg_a - (s - a)n_a) \quad (*) \end{aligned}$$

Now, $(sg_a + (s - a)n_a)(sg_a - (s - a)n_a) = s^2 g_a^2 - (s - a)^2 n_a^2 \stackrel{\text{Bogdan Fusteï}}{=} (s^2((s - a)^2 + 2rh_a) - (s - a)^2(s^2 - 2h_a r_a)) = 2rs^2 h_a + 2h_a(s - a)^2 \cdot \frac{rs}{s - a}$

$= 2rsh_a(s + s - a) = 2rs \cdot \frac{2rs}{a} \cdot (b + c) = 4r^2 s^2 \left(\frac{b+c}{a}\right) \Rightarrow (*)$ is true

$\therefore \frac{2(h_a - r)}{g_a - n_a + \sqrt{4r^2 + (b - c)^2}} + \frac{h_a}{n_a + s} = \frac{g_a + s - a}{2r}$ and analogs

$$\begin{aligned} \Rightarrow 2 \sum_{\text{cyc}} \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b - c)^2}} + \sum_{\text{cyc}} \frac{h_a}{n_a + s} &= \frac{1}{2r} \cdot \sum_{\text{cyc}} g_a + \frac{1}{2r} \sum_{\text{cyc}} (s - a) \\ &= \frac{1}{2r} \cdot \sum_{\text{cyc}} g_a + \frac{s}{2r} \quad \text{and so, } 2 \sum_{\text{cyc}} \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b - c)^2}} + \sum_{\text{cyc}} \frac{h_a}{n_a + s} = \\ & \quad \frac{g_a + g_b + g_c + s}{2r} \quad \forall \Delta ABC \text{ (QED)} \end{aligned}$$