Further Studies on Fermat-Torricelli and Napoleon Points of a Triangle using

PLAGIOGONAL Approach

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Abstract: In this present work, we have attempted to extensively study the geometric characteristics of *Fermat*'s Second point (**X_{14}**) and Second *Napoleon* point (**X_{18}**) using a set of inclined axes, also known as the PLAGIOgonal Axes. As a further continuation of our earlier work similar to the present one, we expanded our discussions on *Fermat*'s First point (**X_{13}**) and First Napoleon point (**X_{17}**) and present some additional relations in this work. In both the cases, we have exemplified our theorems using some challenging geometrical problems available in the social media. We have also tried to make a comparison of the relations between the *Fermat* and *Napoleon* points.

Keywords: *First and Second Fermat – Torricelli points*, **X_{13}**, **X_{14}**, *Napoleon points*, **X_{17}**, **X_{18}**, *PLAGIOGONAL Axes*, *isogonic point*, *ETC*

1.0 Introduction

In Euclidean Plane Geometry, if three *equilateral* triangles are constructed outwardly and also inwardly on the three sides of a scalene triangle no angles of which are greater than 120^0, and then three lines are drawn from each of the vertices of the triangle to the opposite vertices of the equilateral triangles, then these lines concur at a point which is known as the First *Fermat-Torricelli* point, denoted henceforth as **F_1** and Second *Fermat* point, denoted henceforth as **F_2** and or simply, *Fermat* points of the given triangle. These *Fermat* points are also the first *isogonic* points (X_{13} and **X_{14}** in Kimberling's *Encyclopedia of Triangle Centers*, *ETC*) for the above mentioned type of triangle. Further, if lines from each of the vertices of the triangle are drawn connecting the centroids of the outwardly and inwardly drawn equilateral triangles then these lines also become concurrent the respective points of concurrency of these lines are termed as the First and Second *Napoleon* points **N_1** and **N_2** or according to Clark Kimberling's *ETC* nomenclature, as **X_{17}** and **X_{18}** respectively.

In this present work, we have expanded our analysis carried out in [1] which only had focussed on **F_1** and **N_1** points.

2.0 Second Fermat or Fermat – Torricelli Point (F₂ or X₁₄) and allied Theorems:

According to the theorem, we know that if for any scalene triangle in which the largest angle not exceeding 120^0, if three *equilateral* triangles are developed *inwardly* then the three lines drawn from the opposite vertices of the triangle joining the vertices of the inward equilateral triangles thus formed, are *concurrent* and the point of the concurrency is called Second *Fermat* or *Fermat – Torricelli* point denoted as **F_2** or, as per *Kimberling*'s nomenclature as **X_{14}**. The following figure shows an arbitrary triangle ***ABC*** and three equilateral triangles ***A_1BC***, ***B_1CA*** and ***C_1AB*** drawn inward. As per the above theorem, lines ***AA_1***, ***BB_1*** and ***CC_1*** are *concurrent* at **F_2** (**X_{14}**).

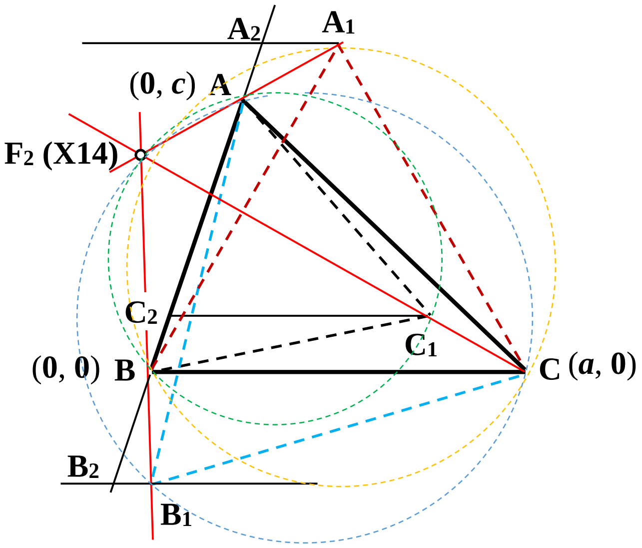

Figure 1: Fermat's Second Point **F**$_2$ (**X**$_{14}$)

Proof:- Let us choose the PLAGIOgonal Co – ordinate system (Vertex, x – axis, y – axis) as (**B**, BC, BA) and it is easy to note the following co-ordinates of the points **A**$_1$, **B**$_1$ and **C**$_1$ respectively as follows:

$$\left(\frac{a\left(\sin B-\sqrt{3}\cos B\right)}{2\sin B},\frac{\sqrt{3}a}{2\sin B}\right),\left(\frac{a\left(\sin B+\sqrt{3}\cos B\right)-\sqrt{3}c}{2\sin B},-\frac{\sqrt{3}a-c\left(\sqrt{3}\cos B+\sin B\right)}{2\sin B}\right)$$

and $\left(\frac{\sqrt{3}c}{2\sin B},\frac{c\sin\left(B-60^0\right)}{\sin B}\right)\Rightarrow$ equations of AA_1, BB_1 and CC_1 are thus respectively:

$$y=c+\frac{\sqrt{3}a-2c\sin B}{a\left(\sin B-\sqrt{3}\cos B\right)}x,\ \ y=-\frac{\sqrt{3}a-c\left(\sqrt{3}\cos B+\sin B\right)}{a\left(\sin B+\sqrt{3}\cos B\right)-\sqrt{3}c}x,\ \ y=\frac{c\left(\sin B-\sqrt{3}\cos B\right)}{\sqrt{3}c-2a\sin B}\left(x-a\right)$$

Solving these equations in pairs we obtain the following same co-ordinates of point of intersection of the above lines as:

$$\boxed{\left(\frac{ca\left(\sin B-\sqrt{3}\cos B\right)\left[a\left(\sin B+\sqrt{3}\cos B\right)-\sqrt{3}c\right]}{2\sqrt{3}\sin B\left[ca\left(\sqrt{3}\sin B+\cos B\right)-c^2-a^2\right]},\frac{ca\left(\sin B-\sqrt{3}\cos B\right)\left[c\left(\sin B+\sqrt{3}\cos B\right)-\sqrt{3}a\right]}{2\sqrt{3}\sin B\left[ca\left(\sqrt{3}\sin B+\cos B\right)-c^2-a^2\right]}\right)}$$

or equivalently:

$$\boxed{\left(\frac{ca\sin\left(B-60^0\right)\left[2a\sin\left(B+60^0\right)-\sqrt{3}c\right]}{\sqrt{3}\sin B\left[2ca\sin\left(B+30^0\right)-c^2-a^2\right]},\frac{ca\sin\left(B-60^0\right)\left[2c\sin\left(B+60^0\right)-\sqrt{3}a\right]}{\sqrt{3}\sin B\left[2ca\sin\left(B+30^0\right)-c^2-a^2\right]}\right)} \qquad \textbf{(1a, 1b)}$$

and *are considered as the co-ordinates of* **F**$_2$ or **X**$_{14}$ *with respect to the above co-ordinate system.*

Alternatively, the above co-ordinates can also be expressed in terms of the sides of the given triangle as follows [eqn (**1c**)]:

$$\boxed{\left(\frac{a}{8\sqrt{3}\Delta}\cdot\frac{\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]\left[4\Delta-\sqrt{3}\left(c^2+b^2-a^2\right)\right]}{\left[4\sqrt{3}\cdot\Delta-\left(a^2+b^2+c^2\right)\right]},\frac{c}{8\sqrt{3}\Delta}\cdot\frac{\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]\left[4\Delta-\sqrt{3}\left(a^2+b^2-c^2\right)\right]}{\left[4\sqrt{3}\cdot\Delta-\left(a^2+b^2+c^2\right)\right]}\right)}$$

Another useful formulation for the co-ordinates of the second *Fermat* point are as follows:

$$\boxed{\left(\frac{2R}{\sqrt{3}}\cdot\frac{4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)}{4\sqrt{3}\Delta-\left(a^2+b^2+c^2\right)}\cdot\sin\left(A-60^0\right),\frac{2R}{\sqrt{3}}\cdot\frac{4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)}{4\sqrt{3}\Delta-\left(a^2+b^2+c^2\right)}\cdot\sin\left(C-60^0\right)\right)} \qquad (\mathbf{1d})$$

[**QED**]

Theorem_II: For any triangle ***ABC*** whose angles are not more than 120^0, if **F2** denotes its second *Fermat* point then prove that:

$$AF_2^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta} \qquad BF_2^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta} \quad \text{and}$$

$$CF_2^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(a^2+b^2-c^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}$$

Proof:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system (Vertex, x – axis, y – axis) as (**B**, ***BC***, ***BA***) and various co-ordinates are marked in the figure itself. For this system, the co-ordinates of **F2** are shown vide **Theorem I** and are expressed in eqn (1a) – (1d). Considering the co-ordinates as expressed in (**1d**), we can prove by *distance formula*: $BF_2^2 = f_1^2+f_2^2+2f_1\cdot f_2\cos B$, where $f_1=\frac{2R}{\sqrt{3}}\cdot\frac{4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)}{4\sqrt{3}\Delta-\left(a^2+b^2+c^2\right)}\cdot\sin\left(A-60^0\right)$ and $f_2=\frac{2R}{\sqrt{3}}\cdot\frac{4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)}{4\sqrt{3}\Delta-\left(a^2+b^2+c^2\right)}\cdot\sin\left(C-60^0\right)$. With a little trigonometric simplification, we can easily prove that: $\boxed{BF_2^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}}$ [**QED**] and the rest two relations can likewise be derived by choosing respectively the PLAGIOgonal system (**A**, ***AB***, ***AC***) and (**C**, ***CB***, ***CA***) [**QED**]

Note: In both the above theorems, Δ represents the area of the triangle ***ABC***.

Theorem_III: For any triangle ***ABC*** whose angles are not more than 120^0, if **F2** denotes its second *Fermat* point then prove that: $AF_2^2+BF_2^2+CF_2^2=\frac{a^2+b^2+c^2}{2}+\frac{2\sqrt{3}\Delta}{3}$

Proof:- Using the Theorem II above, we can easily note that:

$$AF_2^2+BF_2^2+CF_2^2=\frac{1}{6}\cdot\frac{\left[\left\{\left[4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)\right]^2+\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]^2+\left[4\Delta-\sqrt{3}\left(a^2+b^2-c^2\right)\right]^2\right\}\right]}{\left[\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta\right]}$$

which can be simplified to:

$$\Rightarrow AF_2^2+BF_2^2+CF_2^2=\frac{9\left(a^4+b^4+c^4\right)-6\left(a^2b^2+b^2c^2+c^2a^2\right)+48\Delta^2-8\sqrt{3}\Delta\left(a^2+b^2+c^2\right)}{6\left[\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta\right]}$$

$$=\frac{\left[\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta\right]^2+4\left[\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2\right]}{6\left[\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta\right]}$$ and ultimately:

$$AF_2^2+BF_2^2+CF_2^2=\frac{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}{6}+\frac{2}{3}\cdot\frac{\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}$$

$$=\frac{\left(a^2+b^2+c^2\right)-8\sqrt{3}\Delta}{6}+\frac{1}{6}\cdot\frac{8\left(a^4+b^4+c^4\right)-8\left(a^2b^2+b^2c^2+c^2a^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}+\frac{2\sqrt{3}\Delta}{3}$$

$$=\frac{1}{6}\left[\left(a^2+b^2+c^2-8\sqrt{3}\Delta\right)+\frac{8\left(a^4+b^4+c^4\right)-8\left(a^2b^2+b^2c^2+c^2a^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}\right]+\frac{2\sqrt{3}\Delta}{3}$$

$$=\frac{1}{6}\left[\frac{\left(a^2+b^2+c^2\right)^2-12\sqrt{3}\Delta\left(a^2+b^2+c^2\right)+96\Delta^2+8\left(a^4+b^4+c^4\right)-8\left(a^2b^2+b^2c^2+c^2a^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}\right]+\frac{2\sqrt{3}\Delta}{3}$$

$$=\frac{1}{6}\left[\frac{\left(a^2+b^2+c^2\right)^2-12\sqrt{3}\Delta\left(a^2+b^2+c^2\right)+2\left(a^4+b^4+c^4\right)+4\left(a^2b^2+b^2c^2+c^2a^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}\right]+\frac{2\sqrt{3}\Delta}{3}$$

$$=\frac{1}{6}\left[\frac{3\left(a^2+b^2+c^2\right)^2-12\sqrt{3}\Delta\left(a^2+b^2+c^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}\right]+\frac{2\sqrt{3}\Delta}{3}$$

$$\Rightarrow \boxed{AF_2^2+BF_2^2+CF_2^2=\frac{a^2+b^2+c^2}{2}+\frac{2\sqrt{3}\Delta}{3}}$$ **[QED]**

3.0 Properties of 2nd Fermat Point, F_2 (X_{14}):

Having found the basic characters of the 2nd Fermat point, let us observe some of its properties:

Property I: If *ABC* is a triangle with $\angle ABC = 60^0$, then we note from eqn (**1b**) that: **F_2** coincides with the vertex **B**. In other words, if any angle of the triangle ***ABC*** becomes 60^0, then *the 2nd Fermat point belongs to the circumcircle of the triangle*.

It is also clear that for an *equilateral* triangle, 2nd Fermat point, **F_2** or **X_{14}** becomes undefined.

Property II: If ABC is a triangle with $\angle ABC = 120^0$, then we note from eqn (**1b**) that the co-ordinates of **F**$_2$ with respect to the PLAGIOgonal system shown above are:

$$\boxed{\left(\frac{c^2a}{c^2+a^2-ca}, \frac{ca^2}{c^2+a^2-ca}\right)}$$

Property III: If ABC is a triangle with $\angle ABC = 120^0$, then the following is true:

$$\boxed{AF_2 + CF_2 - BF_2 = \sqrt{c^2+a^2-ca} = \sqrt{\frac{a^2+b^2+c^2}{2} - 2\sqrt{3}\Delta}}$$

This can be directly checked if we note that for $\angle ABC = 120^0$ the following results hold good: $AF_2 = \frac{c^2}{\sqrt{c^2+a^2-ca}}, BF_2 = \frac{ca}{\sqrt{c^2+a^2-ca}}, CF_2 = \frac{a^2}{\sqrt{c^2+a^2-ca}}$. So it is obvious that: $AF_2 + CF_2 > BF_2$

Property IV: If ABC is a triangle with $\angle ABC = 120^0$, then ***BF***$_2$ is the *Geometric Mean* (**GM**) of ***AF***$_2$ and ***CF***$_2$

The fact follows easily from **Property III** mentioned above.

Based on the above properties related with **F**$_2$ and a special triangle ABC is a triangle with $\angle ABC = 120^0$, following examples can be discussed in short.

Example # 1: Consider a non-isosceles triangle ABC with $\angle BAC = 120^0$ and whose *circumcenter*, *orthocentre* and the 2nd *Fermat* point and the middle point of the side BC be denoted as **O**, **H**, **F** and **M** respectively . Let the '**A**' – angle bisector meets the circumcircle of the triangle at the point **K**. Prove that $AHOK$ is a *parallelogram* and its area is given as $AM \cdot OF$. [**Problem Courtesy**: *Rachid Iksi*]

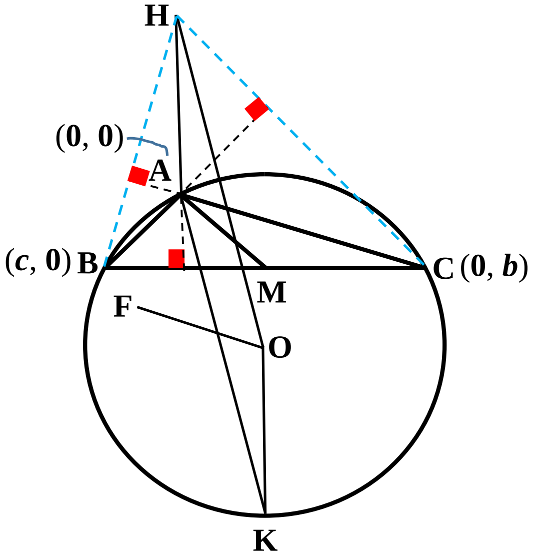

Figure 2: Example #1

Proof:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**A**, ***AB***, ***AC***) and various co-ordinates are marked in the figure itself. We note that the co-ordinates of **O**, **H**, **K** and **F** for the given triangle are respectively: $\left(\frac{b+2c}{3},\frac{2b+c}{3}\right),\left(-\frac{2b+c}{3},-\frac{b+2c}{3}\right),(b+c,b+c),\left(\frac{b^2c}{b^2+c^2-bc},\frac{bc^2}{b^2+c^2-bc}\right)\Rightarrow$ now by distance formula we easily obtain:

$$OH=AK=b+c,\ AH=\sqrt{\frac{b^2+c^2+bc}{3}}=\frac{a}{\sqrt{3}},\ OK=R=\frac{a}{2\sin A}=\frac{a}{\sqrt{3}}\Rightarrow AH=OK$$

(alternatively, we can show that: $AK\parallel OH$ and $OK\parallel HA$) and hence ***AHOK*** *is a parallelogram*.

Further: $$[AHOK]=2[\Delta AOK]=\frac{\sqrt{3}}{2}\begin{vmatrix}0 & 0 & 1\\ b+c & b+c & 1\\ \frac{b+2c}{3} & \frac{2b+c}{3} & 1\end{vmatrix}=\frac{1}{2\sqrt{3}}\left|b^2-c^2\right| \qquad (1)$$

Also we note that: $AM^2=\frac{2\left(b^2+c^2\right)-a^2}{4}\Rightarrow AM=\frac{\sqrt{b^2+c^2-bc}}{2}$ since: $a^2=b^2+c^2+bc$

and:

$$OF^2=\left(\frac{b^2c}{b^2+c^2-bc}-\frac{b+2c}{3}\right)^2+\left(\frac{bc^2}{b^2+c^2-bc}-\frac{2b+c}{3}\right)^2-\left(\frac{b^2c}{b^2+c^2-bc}-\frac{b+2c}{3}\right)\left(\frac{bc^2}{b^2+c^2-bc}-\frac{2b+c}{3}\right)$$

$$=\frac{\left(b^2-c^2\right)^2}{9\left(b^2+c^2-bc\right)^2}\left[(2c-b)^2+(c-2b)^2+(2c-b)(2b-c)\right]=\frac{\left(b^2-c^2\right)^2}{3\left(b^2+c^2-bc\right)}\Rightarrow OF=\frac{\left|b^2-c^2\right|}{\sqrt{3\left(b^2+c^2-bc\right)}}$$

$$\Rightarrow AM\cdot OF=\frac{\left|b^2-c^2\right|}{2\sqrt{3}} \qquad (2)$$

From (1) and (2): $\boxed{[AHOK]=AM\cdot OF}$ [**QED**]

Example # 2: Consider a non-isosceles triangle ***ABC*** with $\angle BAC=120^0$. **F** is the 2nd *Fermat* point, and ***AS*** is the '**A**' – *symmedian* of the given triangle. Prove that: **S** lies on the '**A**' – *symmedian* and further $\frac{1}{AS}\cdot\frac{1}{AF}=\frac{1}{AB^2}+\frac{1}{AC^2}$ [**Problem Courtesy**: *Rachid Iksi*]

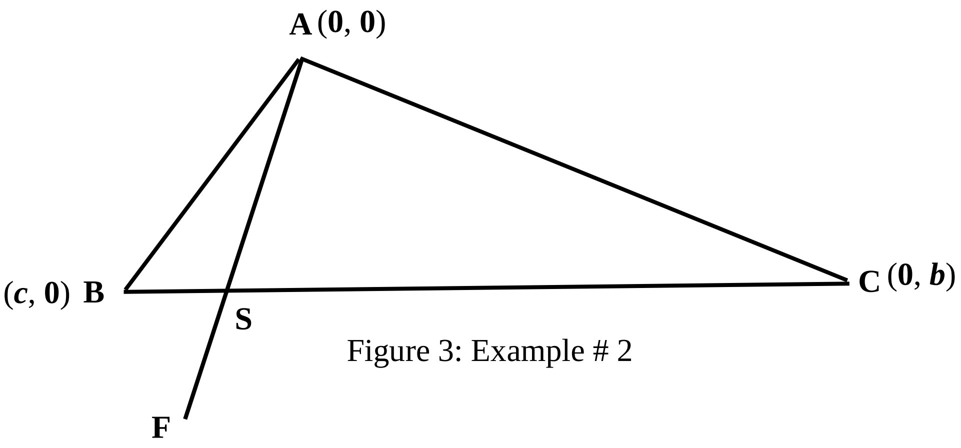

Figure 3: Example # 2

Proof:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**A**, AB, AC) and various co-ordinates are marked in the figure itself. We note that the co-ordinates of **F** for the given triangle are: $\left(\frac{b^2c}{b^2+c^2-bc}, \frac{bc^2}{b^2+c^2-bc}\right) \Rightarrow$ equation of AF is: $y=\frac{c}{b}x \Rightarrow$ **S** lies on the '**A**' – *symmedian* [**QED**]

Now, we know that by *distance formula*: $AS=\frac{bc}{b^2+c^2}\sqrt{b^2+c^2-bc}$ and also by **Theorem – II** we know that: $AF^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}$ which for $\angle BAC=120^0$ becomes:

$$AF=\frac{bc}{\sqrt{b^2+c^2-bc}} \Rightarrow \frac{1}{AS}\cdot\frac{1}{AF}=\frac{b^2+c^2}{b^2c^2} \Rightarrow \boxed{\frac{1}{AS}\cdot\frac{1}{AF}=\frac{1}{AB^2}+\frac{1}{AC^2}}$$

[**QED**]

Example # 3: Consider a non-isosceles triangle ABC with $\angle BAC=120^0$. Prove that its *Lester* Circle (**CL**), i.e., the circle which passes through the circumcenter **O**, center of *NPC* (**CN**), **O9** and *Fermat* points **F1** and **F2** and the *NPC* (**CN**) of the triangle, are congruent. Also prove that (**CN**) and (**CL**) intersect at the points **D** and **E** which are the middle points of AB and AC.

[**Problem Courtesy**: *Rachid Iksi*]

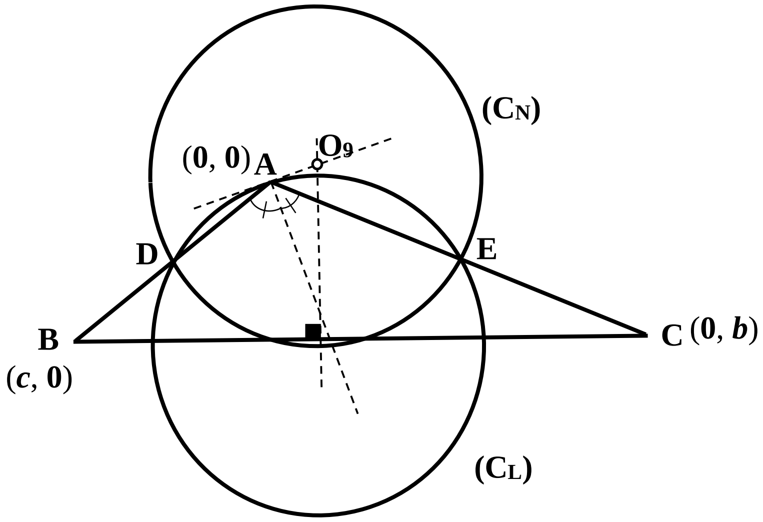

Figure 3: Example #3

Proof:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**A**, AB, AC) and various co-ordinates are marked in the figure itself. We note that the co-ordinates of circumcenter **O** and **O9** are respectively (for $\angle BAC=120^0$) : $\left(\frac{b+2c}{3}, \frac{2b+c}{3}\right)$ and $\left(-\frac{b-c}{6}, \frac{b-c}{6}\right) \Rightarrow m_{OO_9}=\frac{\frac{4b+2c}{6}-\frac{b-c}{6}}{\frac{2b+4c}{6}+\frac{b-c}{6}}=1$. Now equation of the perpendicular bisector of the line joining **O** and $\mathbf{O_9}$ is: $x+y=\frac{b+c}{2}$ (**1**) We now note that the first *Fermat* point, $\mathbf{F_1}$ for this given triangle coincides with the vertex **A** and the co-ordinates of the second *Fermat* point $\mathbf{F_2}$ are: $\left(\frac{b^2c}{c^2+b^2-bc},\frac{bc^2}{c^2+b^2-bc}\right)\Rightarrow m_{F_1F_2}=\frac{c}{b}$ and the '***m***' value of the line which is perpendicular to the line joining the points $\mathbf{F_1}$ and $\mathbf{F_2}$ is given by: $\left(m_{F_1F_2}\right)_\perp=-\frac{1+m_{F_1F_2}\cos A}{m_{F_1F_2}+\cos A}=\frac{2b-c}{b-2c}$ and hence the equation of the perpendicular bisector of $\mathbf{F_1}$ and $\mathbf{F_2}$ is: $(b-2c)y=-bc+(2b-c)x$. Solving it with (**1**) gives us the co-ordinates of the center $\mathbf{O_L}$ of ($\mathbf{C_L}$) as: $\left(\frac{b+2c}{6},\frac{2b+c}{6}\right)$ and as a result, we get the radius $\boldsymbol{R_L}$ of the *Lester* circle as: $R_L^2=\frac{1}{36}\left[(b+2c)^2+(2b+c)^2-(b+2c)(2b+c)\right]=\frac{b^2+c^2+bc}{12}\Rightarrow\boxed{R_L=\frac{R}{2}}$ (since we have the results: $\angle BAC=120^0\Rightarrow a^2=b^2+c^2+bc,\ R=\frac{a}{\sqrt{3}}\Rightarrow 3R^2=b^2+c^2+bc$), ***R*** being the circumradius of the given triangle and since this is also the radius of *NPC*, so prove that for the given triangle, *Lester* Circle ($\mathbf{C_L}$) and the *NPC* of the triangle ($\mathbf{C_N}$) are *congruent* [**QED**]

Now, equation of ($\mathbf{C_L}$) is: $\left(x-\frac{b+2c}{6}\right)^2+\left(y-\frac{2b+c}{6}\right)^2-\left(x-\frac{b+2c}{6}\right)\left(y-\frac{2b+c}{6}\right)=\frac{b^2+c^2+bc}{12}$

$\Rightarrow x^2+y^2-xy-\frac{c}{2}x-\frac{b}{2}y=0\Rightarrow$ it intersects the sides ***AB*** and ***AC*** at points **D** and **E** which are clearly the middle points of ***AB*** and ***AC*** respectively and thus lies on ($\mathbf{C_N}$) and hence we conclude: ($\mathbf{C_N}$) and ($\mathbf{C_L}$) intersect at the points **D** and **E** which are the middle points of ***AB*** and ***AC*** [**QED**]

<u>Example</u> # 4: Consider a non-isosceles triangle ***ABC*** with $\angle BAC=120^0$. ***AD*** is the angle – bisector, **Le** is the *Lemonie* point, **O** is the *circumcenter*, **H** is the *orthocentre*, **G** is the *centroid*, $\mathbf{O_9}$ is the center of *NPC* and $\mathbf{F_2}$ is the 2nd *Fermat* point of the triangle ***ABC***. $\mathbf{M_a}$, $\mathbf{M_b}$ and $\mathbf{M_c}$ are the middle points of the sides ***BC***, ***CA*** and ***AB*** respectively. **N** and **K** are respectively the middle points of ***AD*** and $\boldsymbol{O_9D}$. **M** is the projection of $\mathbf{O_9}$ on $\boldsymbol{M_bM_c}$ and **L** is the reflection of $\mathbf{O_9}$ with respect to **M**. **E** is defined as the second point of intersection of ***OG*** with $\mathbf{C_1}$. Let us define the following system of circles:

$\mathbf{C_1}$: $\odot(G,GO)$ $\mathbf{C_2}$: $\odot(D,\ DA)$ $\mathbf{C_3}$: *Lester* Circle (**Example # 3**)

$\mathbf{C_4}$: Circle through **A**, $\mathbf{O_9}$, **V** and **D** $\mathbf{C_5}$: *NPC* $\mathbf{C_6}$: *Circumcircle* of triangle ***ABC***

<u>Prove that</u>: 1) **L** belongs to *NPC* ($\mathbf{C_5}$) 2) $\mathbf{M_b}$, $\mathbf{M_c}$, **M** and **N** are *collinear*

3) **O**, **G**, $\mathbf{O_9}$ and **H** are *collinear* 4) **E**, **A**, **Le** and $\mathbf{F_2}$ are *collinear*

5) **P**, **R**, **N**, **K**, **S** and **Q** are *collinear* (**R** and **S** are on ***AB*** and ***AC*** respectively)

6) $\boldsymbol{LM_cO_9M_b}$ and ***DRAS*** are *rhombuses* ($\mathbf{60^0}$ – $\mathbf{120^0}$ – $\mathbf{60^0}$ – $\mathbf{120^0}$)

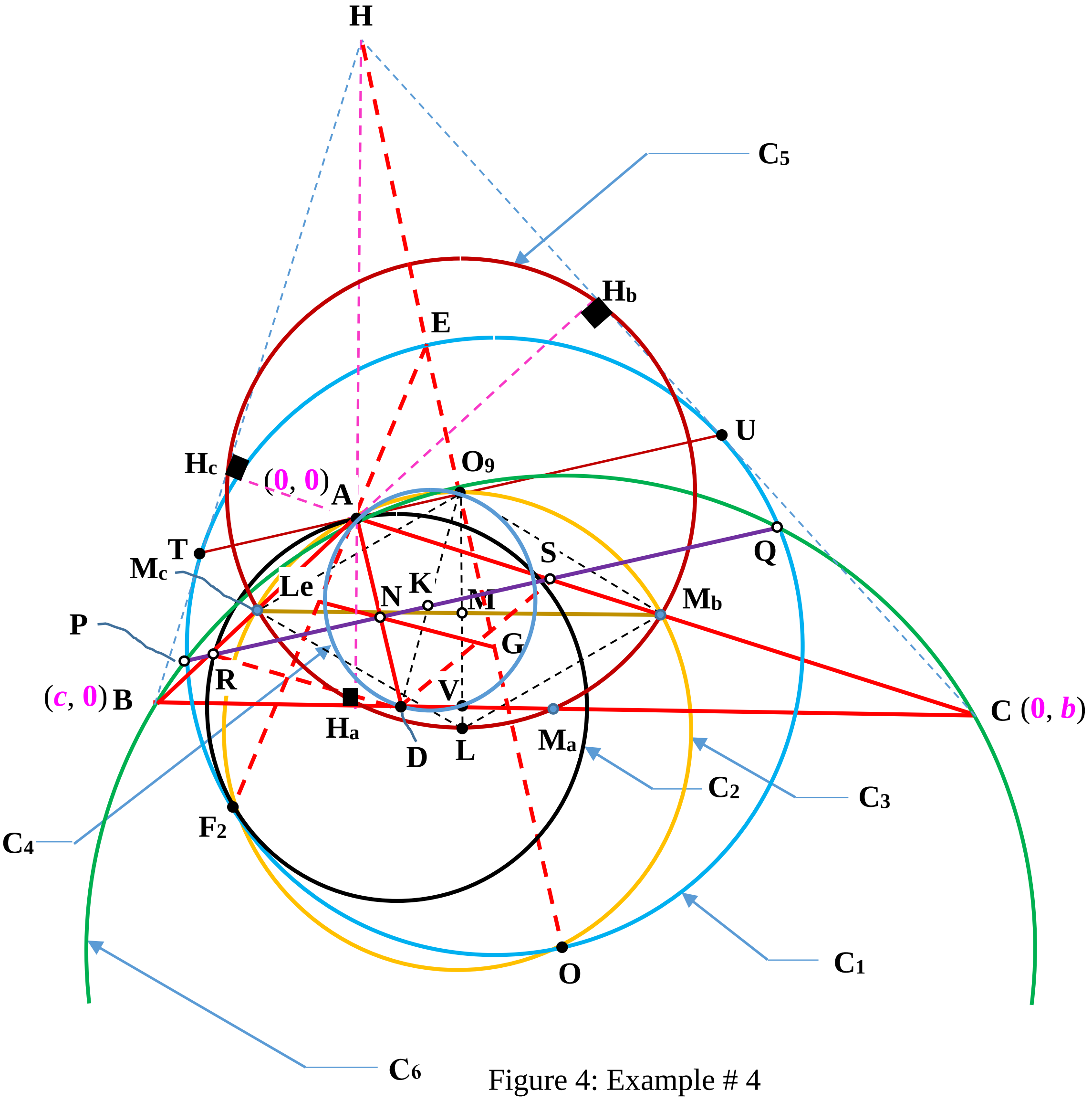

Figure 4: Example # 4

7) ***BH*** and ***CH*** are tangents to $\mathbf{C_1}$ (T and U are touch points)

8) **T**, **A**, $\mathbf{O_9}$ and **U** are *collinear*

[**Problem Courtesy**: *Thanasis Gakopoulos*]

Proof:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**A**, ***AB***, ***AC***) and various co-ordinates are marked in the figure itself. We note that the equations of various circles ($\mathbf{C_1}$ through $\mathbf{C_6}$ except $\mathbf{C_4}$) are respectively as follows for the given triangle: $C_1 : x^2 + y^2 - xy + \left(\frac{b-2c}{3}\right)x - \left(\frac{2b-c}{3}\right)y - \frac{bc}{3} = 0$

$$C_2 : x^2 + y^2 - xy - \left(\frac{bc}{b+c}\right)(x+y) = 0$$

$C_3 : x^2 + y^2 - xy - \frac{c}{2}x - \frac{b}{2}y = 0,\ C_5 : x^2 + y^2 - xy + (x-y)\left(\frac{b-c}{2}\right) = 0$ and finally,

$C_6 : x^2 + y^2 - xy - cx - by = 0$. Now, equation of M_bM_c is: $bx + cy = \frac{bc}{2} \Rightarrow$ co-ordinates of **M** are: $\left(\frac{c}{4}, \frac{b}{4}\right) \Rightarrow$ co-ordinates of **L** are: $\left(\frac{b+2c}{6}, \frac{2b+c}{6}\right)$

$$\Rightarrow O_9L^2 = \frac{1}{36}\left[(2b+c)^2 + (b+2c)^2 - (2b+c)(b+2c)\right] = \frac{b^2+c^2+bc}{12}$$

$$\Rightarrow O_9L^2 = \frac{R^2}{4} = R_9^2 \Rightarrow \boxed{L \in C_5}$$ [QED]

Clearly, co-ordinates of **N** and **M** are $\left(\frac{bc}{2(b+c)}, \frac{bc}{2(b+c)}\right), \left(\frac{c}{4}, \frac{b}{4}\right)$ respectively and they satisfy the equation of M_bM_c and hence: **M_b**, **M_c**, **M** and **N** are collinear [**QED**]

As the co-ordinates of **O**, **G** and **O₉** are: $\left(\frac{b+2c}{3}, \frac{2b+c}{3}\right), \left(\frac{c}{3}, \frac{b}{3}\right), \left(-\frac{b-c}{6}, \frac{b-c}{6}\right)$

$$\Rightarrow m_{OG} = 1, m_{O_9G} = \frac{\frac{b-c}{6} - \frac{2b}{6}}{-\frac{b-c}{6} - \frac{2c}{6}} = 1 \Rightarrow O - G - O_9 \parallel AD \Rightarrow$$ equation of this line is: $x - y + \frac{b-c}{3} = 0$ and it is clear that the co-ordinates of **H** — the *orthocentre* of the triangle $\left(-\frac{2b+c}{3}, -\frac{b+2c}{3}\right)$ satisfy this equation and thus: **O**, **G**, **O₉** and **H** are *collinear* [**QED**]

If we now solve the above line with **C₁**, we obtain the co-ordinates of the point **E** as: $\left(-\frac{b}{3}, -\frac{c}{3}\right) \Rightarrow E \in y = \frac{c}{b}x \Rightarrow E - Le - A - F_2$ are *collinear* [**QED**]

We now note that the line PQ is the *common chord* between the circles **C₁** and **C₆** and hence its equation is: $x + y = \frac{bc}{b+c}$ which can be obtained by subtracting the two equations of the circles. Now it intersects the sides AB and AC of the triangle at points **R** and **S** respectively having the co-ordinates $\left(\frac{bc}{b+c}, 0\right)$ and $\left(0, \frac{bc}{b+c}\right) \Rightarrow ARDS$ is a rhombus formed with two equilateral triangles ADR and ADS [**QED**]

We note the co-ordinates of the points **N** (middle point of AD) and **K** (middle point of O_9D) as:
$\left(\frac{bc}{2(b+c)}, \frac{bc}{2(b+c)}\right), \left(\frac{6bc - b^2 + c^2}{12(b+c)}, \frac{6bc + b^2 - c^2}{12(b+c)}\right) \Rightarrow N, K \in PQ : x + y = \frac{bc}{b+c} \Rightarrow P, R, N, K, S, Q$ are *collinear* [**QED**]

We again note that $O_9L^2 = \frac{b^2+c^2+bc}{12}, O_9M_c^2 = \left(\frac{c}{2}+\frac{b-c}{6}\right)^2 + \frac{(b-c)^2}{36} + \left(\frac{c}{2}+\frac{b-c}{6}\right)\frac{b-c}{6}$ $\Rightarrow O_9M_c^2 = \frac{1}{36}\left[(b+2c)^2+(b-c)^2+(b+2c)(b-c)\right] = \frac{b^2+c^2+bc}{12}$ Similarly we obtain by *distance formula*:

$LM_c^2 = \frac{b^2+c^2+bc}{12}$, $O_9M_b^2 = \frac{(b-c)^2}{36} + \left(\frac{b}{2}-\frac{b-c}{6}\right)^2 - \frac{b-c}{6}\left(\frac{b}{2}-\frac{b-c}{6}\right) = \frac{b^2+c^2+bc}{12}$ and finally:

$LM_b^2 = \left(\frac{b+2c}{6}\right)^2 + \left(\frac{2b+c}{6}-\frac{b}{2}\right)^2 - \left(\frac{b+2c}{6}\right)\left(\frac{2b+c}{6}-\frac{b}{2}\right) = \frac{b^2+c^2+bc}{12}$ and thus we conclude that $LM_cO_9M_b$ and ***DRAS*** are *rhombuses* (**60^0 – 120^0 – 60^0 – 120^0**) [**QED**]

Finally we consider that the equation of the tangents that can be drawn from the point **H** $\left(-\frac{2b+c}{3}, -\frac{b+2c}{3}\right)$ to the circle **C**$_1$ can be written as: $mx - y + \frac{b(2m-1)+c(m-2)}{3} = 0$ and since it is tangent to **C**$_1$ whose center is: $\left(\frac{c}{3}, \frac{b}{3}\right)$ and therefore:

$$\frac{\frac{1}{9}\left[mc-b+b(2m-1)+c(m-2)\right]\frac{3}{4}}{m^2-m+1} = \frac{(b+c)^2}{9} \Rightarrow \frac{1}{3}\cdot\frac{(b+c)^2(m-1)^2}{m^2-m+1} = \frac{(b+c)^2}{9}$$

$\Rightarrow m = 2 \wedge \frac{1}{2} \Rightarrow T_1 : \boxed{2x-y+b=0}$ —— which is the equation of ***HC*** for $m=2$ similarly we can find the equation of the second tangent $\left(m=\frac{1}{2}\right)$ as: $T_2 : \frac{x}{2} - y - \frac{c}{2} = 0 \Rightarrow \boxed{y = \frac{x-c}{2}}$ —— which is the equation of ***HB*** and the co-ordinates of the touch points are easily found to be as **T** and **U** having the co-ordinates: $\left(\frac{c}{3}, -\frac{c}{3}\right)$ and $\left(\frac{b}{3}, -\frac{b}{3}\right)$ satisfying the equation: $x+y=0$ and thus we note that **T**, **A**, **O**$_9$ and **U** are *collinear* [**QED**]

<u>Example</u> # 5: Consider a non-isosceles triangle ***ABC*** with $\angle ABC = 45^0, \angle ACB = 30^0$. **O** and **F**$_2$ are the *circumcenter* and the second *Fermat* point respectively while ***DEF*** is the *orthic* triangle of ***ABC***. Prove that $\Delta^s DEF, ABF_2, AOF_2$ are **30^0 – 60^0 – 90^0** triangles. Also show that: $\frac{[\Delta DEF]}{[\Delta AOF_2]} = \frac{[\Delta DEF]}{[\Delta ABF_2]} = \frac{3}{4}$ [**Problem Courtesy**: *Rachid Iksi*]

<u>Proof</u>:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**C**, ***CA***, ***CB***) and various co-ordinates are marked in the figure itself. Now for the given condition $\angle ABC = 45^0, \angle ACB = 30^0$ we note the following:

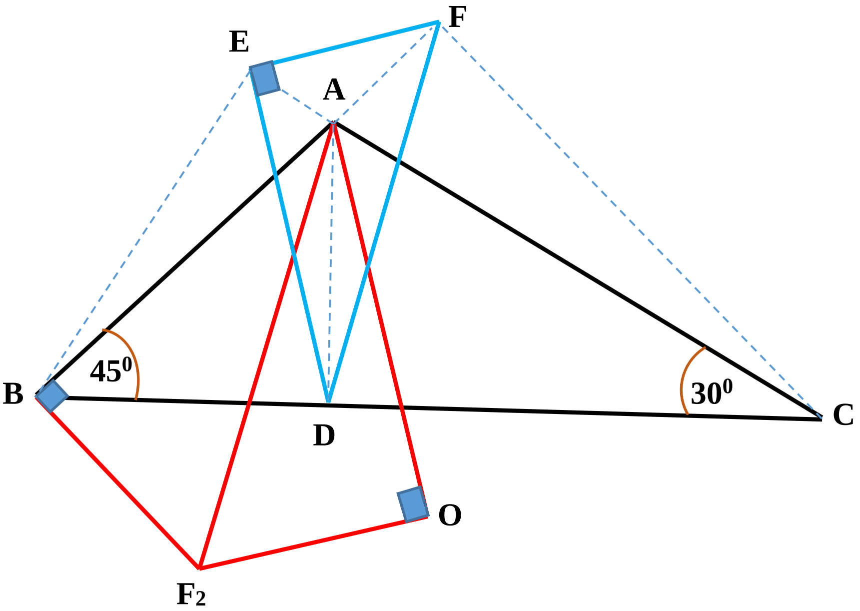

Figure 5: Example # 5

$\frac{a}{\sin 75^0}=\frac{b}{\sin 45^0}=\frac{c}{\sin 30^0}\Rightarrow \frac{a}{\sqrt{3}+1}=\frac{b}{2}=\frac{c}{\sqrt{2}}\Rightarrow a=\left(\sqrt{3}+1\right), b=2, c=\sqrt{2}$ (WLOG). Hence, we note that the co-ordinates of the points **O**, **F_2**, **D**, **E** and **F** are respectively given as follows: $\left(1-\sqrt{3}, 2\right), \left(-\frac{2}{\sqrt{3}}, \frac{2\left(\sqrt{3}+1\right)}{\sqrt{3}}\right), \left(0, \sqrt{3}\right), \left(\frac{\sqrt{3}\left(\sqrt{3}+1\right)}{2}, 0\right), \left(\sqrt{3}+1, -1\right)\Rightarrow$ we note that:

$m_{CF}=-\frac{\sqrt{3}-1}{2},\quad m_{BF_2}=-\frac{\sqrt{3}-1}{2}\Rightarrow CF \parallel BF_2 \Rightarrow BF_2 \perp AB$. Also from our known results:

$$AF_2=\frac{1}{\sqrt{6}}\cdot\frac{4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)}{\sqrt{a^2+b^2+c^2-4\sqrt{3}\Delta}}=\frac{1}{\sqrt{6}}\cdot\frac{2\sqrt{3}+2-2\sqrt{3}\left(1-\sqrt{3}\right)}{2}=\frac{1}{\sqrt{6}}\cdot\frac{8}{2}=\frac{4}{\sqrt{6}}$$ and

$$BF_2=\frac{1}{\sqrt{6}}\cdot\frac{\left|4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right|}{\sqrt{a^2+b^2+c^2-4\sqrt{3}\Delta}}=\frac{1}{\sqrt{6}}\cdot\frac{\left|2\sqrt{3}+2-2\sqrt{3}\left(\sqrt{3}+1\right)\right|}{2}=\frac{2}{\sqrt{6}}$$ Thus:

$\Rightarrow \tan\left(\angle BAF_2\right)=\frac{BF_2}{AB}=\frac{1}{\sqrt{3}}\Rightarrow \angle BAF_2=30^0\Rightarrow$ triangle ABF_2 is a $30^0-60^0-90^0$ triangle

Again we note that: $m_{AO}=\frac{2a-\sqrt{3}b}{b-\sqrt{3}a}=-\frac{2}{\sqrt{3}+1}=-\left(\sqrt{3}-1\right)$ and also we have:

$$m_{OF_2}=\frac{\frac{2\left(\sqrt{3}+1\right)}{\sqrt{3}}-2}{-\frac{2}{\sqrt{3}}-1+\sqrt{3}}=\frac{\frac{2}{\sqrt{3}}}{\frac{1-\sqrt{3}}{\sqrt{3}}}=-\left(\sqrt{3}+1\right)\Rightarrow 1+m_{AO}\cdot m_{OF_2}+\left(m_{AO}+m_{OF_2}\right)\cos C=1+2-3=0\Rightarrow AO\perp OF_2$$

Thus ***AOF_2*** is right angled triangle and further we note that:

$R=\frac{AB}{2\sin 30^0}=AB\Rightarrow \Delta ABF_2\cong \Delta AOF_2\Rightarrow$ triangle AOF_2 is a $\mathbf{30^0-60^0-90^0}$ triangle.

Finally: $m_{DE}=-\frac{2}{\sqrt{3}+1}=-\left(\sqrt{3}-1\right)\Rightarrow DE\parallel AO$ and also we note that:

$m_{EF}=\frac{2}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-2\right)}=-\frac{2\left(\sqrt{3}+2\right)}{\sqrt{3}+1}=-\left(\sqrt{3}+1\right)\Rightarrow EF\parallel OF_2\Rightarrow EF\perp DE\Rightarrow$ DEF is a right angled triangle and due to the above parallelism we note that ΔAOF_2 is a $\mathbf{30^0-60^0-90^0}$ triangle. **[QED]**

Now we note by *distance formula*: $DF=\sqrt{2\left(\sqrt{3}+1\right)^2-\sqrt{3}\left(\sqrt{3}+1\right)^2}=\sqrt{2}$ and since

$$\Delta AOF_2\sim\Delta DEF_2\Rightarrow\frac{\left[\Delta DEF_2\right]}{\left[\Delta AOF_2\right]}=\frac{DF_2^2}{AF_2^2}=\frac{6}{4}\cdot\frac{1}{2}=\frac{3}{4}\Rightarrow\boxed{\frac{\left[\Delta DEF_2\right]}{\left[\Delta AOF_2\right]}=\frac{\left[\Delta DEF_2\right]}{\left[\Delta ABF_2\right]}=\frac{3}{4}}\quad \text{[QED]}$$

4.0 Comparison between 1st and 2nd Fermat Points, F1, F2 (X13, X14):

In our earlier work [1], we had discussed the properties of the First *Fermat* point (also known as the *Fermat – Torricelli* point), **F1** or **X13**. In this present section, we will discuss the similarity of the two *Fermat* points **F1** and **F2**. The following figure shows the **F1** point for any triangle ***ABC***.

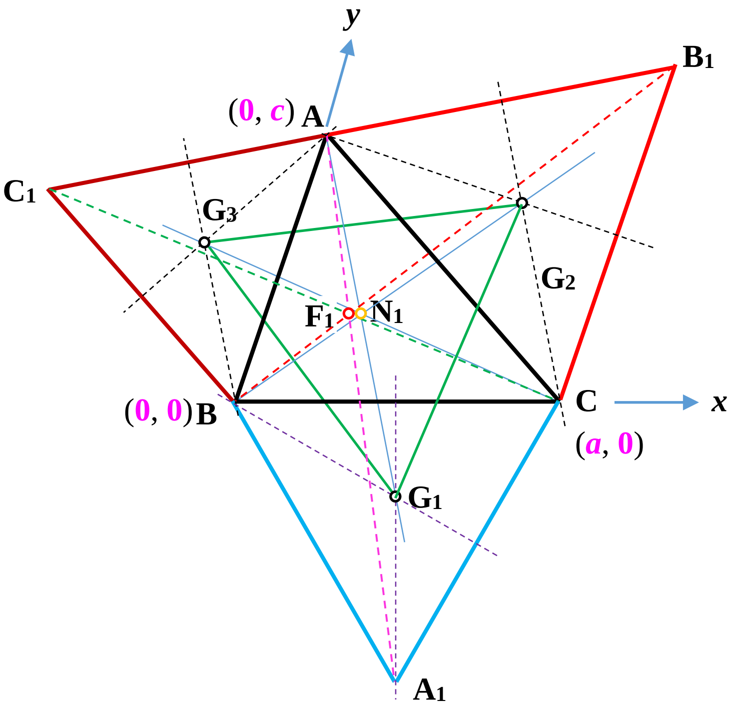

Figure 6: Fermat's First Point (**F1**) and Napoleon's First Point (**N1**)

a) Co-ordinates of **F1** and **F2:**

We list the following co-ordinates with respect to the PLAGIOgonal Co – ordinate system) as (**B**, ***BC***, ***BA***) and various co-ordinates are marked accordingly in the figure itself.

Co-ordinates of **F1**: $$\left(\frac{ca\sin\left(B+60^0\right)\left[2a\sin\left(60^0-B\right)-\sqrt{3}c\right]}{\sqrt{3}\sin B\left[2ca\sin\left(30^0-B\right)-a^2-c^2\right]},\frac{ca\sin\left(B+60^0\right)\left[2c\sin\left(60^0-B\right)-\sqrt{3}a\right]}{\sqrt{3}\sin B\left[2ca\sin\left(30^0-B\right)-a^2-c^2\right]}\right)$$

and:

Co-ordinates of **F2**: $$\left(\frac{ca\sin\left(B-60^0\right)\left[2a\sin\left(60^0+B\right)-\sqrt{3}c\right]}{\sqrt{3}\sin B\left[2ca\sin\left(30^0+B\right)-c^2-a^2\right]},\frac{ca\sin\left(B-60^0\right)\left[2c\sin\left(60^0+B\right)-\sqrt{3}a\right]}{\sqrt{3}\sin B\left[2ca\sin\left(30^0+B\right)-c^2-a^2\right]}\right)$$

Hence from the above it is obvious that if *any one angle of otherwise a scalene triangle is* $\mathbf{120^0}$, *then* ***F1*** *will be on its circumcircle* while, if *any one angle of otherwise a scalene triangle is* $\mathbf{60^0}$*, then* ***F2*** *will be on its circumcircle.*

b) Distances of **F1** and **F2** from the Triangle Vertices**:**

We note from the above co-ordinates of **F1** and **F2** that:

$$AF_1^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(b^2+c^2-a^2\right)+4\Delta\right]^2}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta}\quad BF_1^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(c^2+a^2-b^2\right)+4\Delta\right]^2}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta}\quad CF_1^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(a^2+b^2-c^2\right)+4\Delta\right]^2}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta}$$

and:

$$AF_2^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(b^2+c^2-a^2\right)-4\Delta\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta},\quad BF_2^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(c^2+a^2-b^2\right)-4\Delta\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta},\quad CF_2^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(a^2+b^2-c^2\right)-4\Delta\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}$$

From the previous set of equations we can conclude that:

$$AF_1^2+BF_1^2+CF_1^2=\frac{a^2+b^2+c^2}{2}-\frac{2\sqrt{3}\Delta}{3}$$ for **F1** and similarly for **F2**, we have:

$$AF_2^2+BF_2^2+CF_2^2=\frac{a^2+b^2+c^2}{2}+\frac{2\sqrt{3}\Delta}{3}$$ which we have proved in **Theorem III** above.

c) Distances of **F1** and **F2** from the *Centroid* of the Triangle**:**

In this case we apply *Leibnitz*' Theorem which states that: $\sum_i MA_i^2=\sum_i GA_i^2+n\cdot MG^2$, where there are ***n*** set of points (**A1**, **A2**, **A3**, ……., **An**) whose centroid is **G** and **M** be any arbitrary point. Taking the point **M** as *Fermat* points, we get the following Theorems quite easily:

$$GF_1^2=\frac{a^2+b^2+c^2}{18}-\frac{2\sqrt{3}\Delta}{9}=\frac{a^2+b^2+c^2-4\sqrt{3}\Delta}{18}$$ for **F1**

and for **F2** we get: $$GF_2^2=\frac{a^2+b^2+c^2}{18}+\frac{2\sqrt{3}\Delta}{9}=\frac{a^2+b^2+c^2+4\sqrt{3}\Delta}{18}$$

We observe lots of similarities between the characteristics of **F₁** and **F₂** as obvious from the above comparative analysis.

We conclude the session by considering the following examples.

5.0 Examples on 1st and 2nd Fermat Points, F₁, F₂ (X₁₃, X₁₄):

<u>Example</u> # 6: Consider a triangle ABC of which $\angle ABC = 60^0$. Prove that for such a triangle, $AF_1^2 + BF_1^2 + CF_1^2 = AC^2$ [**Problem Courtesy**: *Rachid Iksi*]

<u>Proof</u>:- We know that: $AF_1^2 + BF_1^2 + CF_1^2 = \frac{a^2+b^2+c^2}{2} - \frac{2\sqrt{3}\Delta}{3}$ and also we note that:

$b^2 = a^2 + c^2 - ca,\ 4\Delta = \sqrt{3}ca \Rightarrow \frac{2\sqrt{3}\Delta}{3} = \frac{ca}{2}$. Now if we substitute these into above relation, we get: $AF_1^2 + BF_1^2 + CF_1^2 = \frac{a^2+b^2+c^2}{2} - \frac{2\sqrt{3}\Delta}{3} = \frac{2(a^2+c^2)-ca}{2} - \frac{ca}{2} = a^2 + c^2 - ca$

$\Rightarrow \boxed{AF_1^2 + BF_1^2 + CF_1^2 = b^2 = AC^2}$ [**QED**]

<u>Example</u> # 7: Consider a triangle ABC of which $\angle ABC = 60^0$ with BS and **F₁** as '**B**' – *symmedian* and *Fermat – Torricelli* point respectively of the triangle . It is also given that BS = CS. Show that: $\frac{BF_1}{AF_1} = \frac{CF_1}{BF_1} = \frac{AF_1 + BF_1 + CF_1}{AC} = \varphi = \frac{\sqrt{5}+1}{2}$. Also prove that: $CF_1 = AF_1 + BF_1$ and $\frac{1}{AF_1} = \frac{1}{BF_1} + \frac{1}{CF_1}$ [**Problem Courtesy**: *Rachid Iksi*]

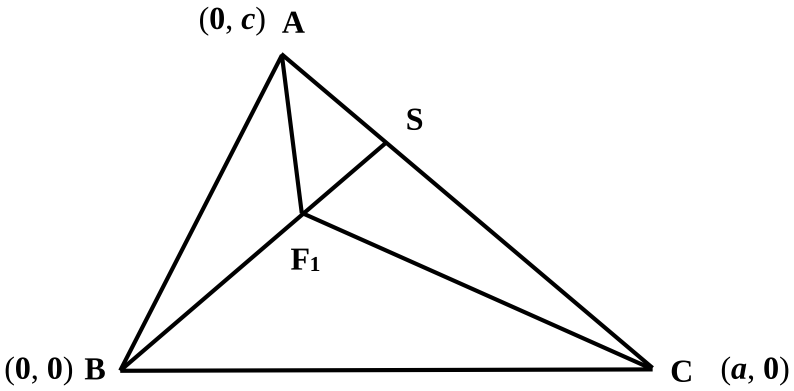

Figure 7: Example # 7

<u>Proof</u>:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**B**, BC, BA) and various co-ordinates are marked in the figure itself. Since $\angle ABC = 60^0$, we note: $\Rightarrow b^2 = a^2 + c^2 - ca,\ 4\Delta = \sqrt{3}ca$. Also: We note that the co-ordinates of **S** are:

$\left(\frac{c^2a}{a^2+c^2},\frac{ca^2}{a^2+c^2}\right)\Rightarrow BS=\frac{ca}{a^2+c^2}\sqrt{a^2+c^2+ca}$ and also: $CS=\frac{ba^2}{a^2+c^2}$ Thus $BS=CS\Rightarrow\frac{ba^2}{a^2+c^2}=\frac{ca}{a^2+c^2}\sqrt{a^2+c^2+ca}\Rightarrow a^2-c^2=ca$

$\Rightarrow\frac{a}{c}=\frac{\sqrt{5}+1}{2}=\varphi$ Also we know that: $\frac{b^2}{c^2}=1+\frac{a^2}{c^2}-\frac{a}{c}=2\Rightarrow\frac{b}{c}=\sqrt{2}$. Now from our foregoing analysis we know that: $\frac{BF_1}{AF_1}=\frac{\sqrt{3}\left(c^2+a^2-b^2\right)+4\Delta}{\sqrt{3}\left(b^2+c^2-a^2\right)+4\Delta}=\frac{c^2+a^2-b^2+ca}{b^2+c^2-a^2+ca}=\frac{a}{c}=\varphi$, $\frac{CF_1}{BF_1}=\frac{a^2+b^2-c^2+ca}{c^2+a^2-b^2+ca}=\frac{a}{c}=\varphi$. Further from [1], it is a known result that:

$AF_1+BF_1+CF_1=\sqrt{\frac{a^2+b^2+c^2+4\sqrt{3}\Delta}{2}}\Rightarrow$ for the present triangle, we have:

$$AF_1+BF_1+CF_1=\sqrt{\frac{a^2+b^2+c^2+3ca}{2}}=\sqrt{a^2+c^2+ca}\Rightarrow\frac{AF_1+BF_1+CF}{AC}=\sqrt{\frac{a^2+c^2+ca}{b^2}}=\sqrt{\frac{1+\frac{a}{c}+\frac{a^2}{c^2}}{\frac{b^2}{c^2}}}$$

$$\Rightarrow\frac{AF_1+BF_1+CF}{AC}=\sqrt{\frac{1+\varphi+\varphi^2}{2}}=\sqrt{\varphi+1}\Rightarrow\frac{AF_1+BF_1+CF}{AC}=\varphi$$

$$\Rightarrow\boxed{\frac{BF_1}{AF_1}=\frac{CF_1}{BF_1}=\frac{AF_1+BF_1+CF}{AC}=\varphi}$$ **[QED]**

Now from the above ratios we get: $CF_1=\varphi BF_1,\ AF_1=\frac{1}{\varphi}\cdot BF_1\Rightarrow CF_1-AF_1=\left(\varphi-\frac{1}{\varphi}\right)BF_1$

$\Rightarrow\boxed{CF_1=AF_1+BF_1}$ **[QED]** Further, we note that: $\frac{1}{AF_1}=\varphi\cdot\frac{1}{BF_1},\ \frac{1}{CF_1}=\frac{1}{\varphi}\cdot\frac{1}{BF_1}$

$\Rightarrow\frac{1}{AF_1}-\frac{1}{CF_1}=\left(\varphi-\frac{1}{\varphi}\right)\frac{1}{BF_1}\Rightarrow\boxed{\frac{1}{AF_1}=\frac{1}{BF_1}+\frac{1}{CF_1}}$ **[QED]**

<u>Example</u> # 8: Consider a triangle ABC of which $\angle ABC=60^0,\angle ACB=45^0$ with BM and **F₁** as '**B**' –*median* and *Fermat – Torricelli* point respectively of the triangle. If R denotes the circumradius of the triangle, then prove that: $R^2=AF_1\cdot BM$ [**Problem Courtesy**: *Rachid Iksi*]

<u>Proof</u>:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**B**, ***BC***, ***BA***) and various co-ordinates are marked in the figure itself. Since $\angle ABC=60^0,\angle ACB=45^0\Rightarrow$ we note: $\frac{a}{\sin 75^0}=\frac{b}{\sin 60^0}=\frac{c}{\sin 45^0}\Rightarrow\frac{a}{\sqrt{3}+1}=\frac{b}{\sqrt{6}}=\frac{c}{2}$. Thus WLOG we assume: $a=\sqrt{3}+1,\ b=\sqrt{6},\ c=2\Rightarrow 2R=\frac{c}{\sin 45^0}=2\sqrt{2}\Rightarrow R=\sqrt{2}$ Also we

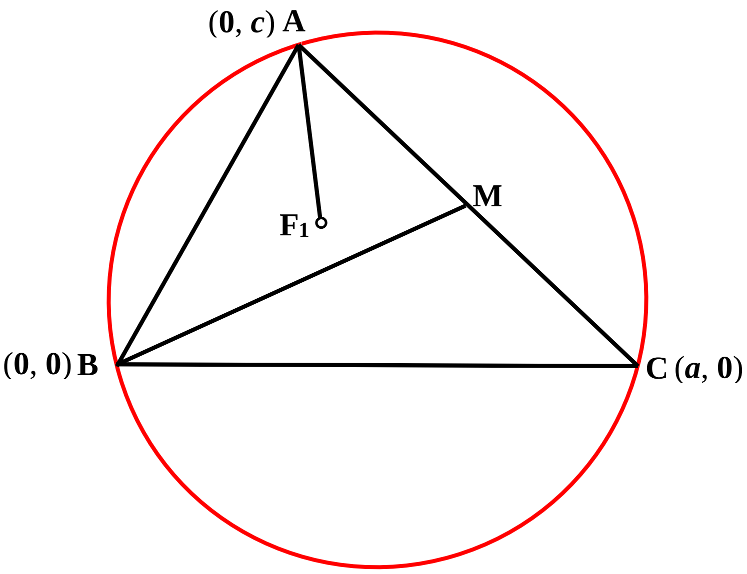

Figure 8: Example # 8

note that: $BM^2=\frac{2\left(a^2+c^2\right)-b^2}{4}\Rightarrow BM=\frac{\sqrt{10+4\sqrt{3}}}{2}$ and also we find from our previous work that:

$$AF_1=\frac{1}{\sqrt{6}}\cdot\frac{\sqrt{3}\left(b^2+c^2-a^2\right)+4\Delta}{\sqrt{4\sqrt{3}\Delta+a^2+b^2+c^2}}=\frac{1}{\sqrt{2}}\cdot\frac{b^2+c^2-a^2+ca}{\sqrt{3ca+a^2+b^2+c^2}}=\frac{c^2}{\sqrt{a^2+c^2+ca}}\Rightarrow AF_1=\frac{4}{\sqrt{10+4\sqrt{3}}}$$

$$\Rightarrow AF_1\cdot BM=\frac{4}{\sqrt{10+4\sqrt{3}}}\cdot\frac{\sqrt{10+4\sqrt{3}}}{2}=\left(\sqrt{2}\right)^2\Rightarrow\boxed{R^2=AF_1\cdot BM}\qquad\text{[QED]}$$

<u>Example</u> # 9: Consider a triangle ***ABC*** of which ***AS*** and ***BT*** are two *symmedian*s such that ***AS*** = ***BS*** and ***BT*** = ***CT***. **F**$_1$ is the *Fermat – Torricelli* point of the triangle. Prove that: $\frac{AC}{AF_1+BF_1+CF_1}=\sqrt{1-\frac{\sqrt{3}}{\sqrt{7}}}$ [**Problem Courtesy**: *Rachid Iksi*]

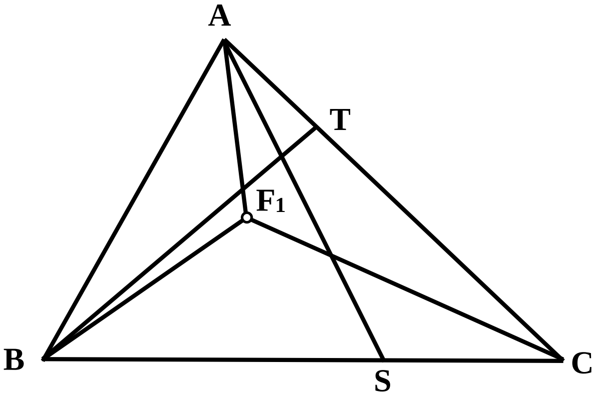

Figure 9: Example # 9

Proof:- From our earlier exercise (Refer to **Example # 7**) we know that $AS = BS \Rightarrow a = \sqrt{2}b$ and similarly, $BT = CT \Rightarrow b = \sqrt{2}c$. Thus we must have: $a = \sqrt{2}b = 2c \Rightarrow \frac{a}{2} = \frac{b}{\sqrt{2}} = \frac{c}{1}$. Thus, WLOG we assume: $a = 2, b = \sqrt{2}, c = 1$. Hence $4\Delta = \sqrt{7}$. Now we note from our analysis [1]:

$$\frac{AC}{AF_1 + BF_1 + CF_1} = \frac{\sqrt{2}b}{\sqrt{a^2 + b^2 + c^2 + 4\sqrt{3}\Delta}} = \frac{2}{\sqrt{7 + \sqrt{21}}} = \frac{\sqrt{7 - \sqrt{21}}}{\sqrt{7}}$$

$$\Rightarrow \boxed{\frac{AC}{AF_1 + BF_1 + CF_1} = \sqrt{1 - \frac{\sqrt{3}}{\sqrt{7}}}} \qquad \text{[QED]}$$

Example # 10: Consider a triangle ***ABC*** of which the largest angle is not more than 120^0 and **G** and **F₁** be its *centroid* and the first *Fermat* point respectively. Prove that: $CF_1 \perp GF_1 \Leftrightarrow c^2 = \frac{a^2 + b^2}{2}$ [**Problem Courtesy**: *Rachid Iksi*]

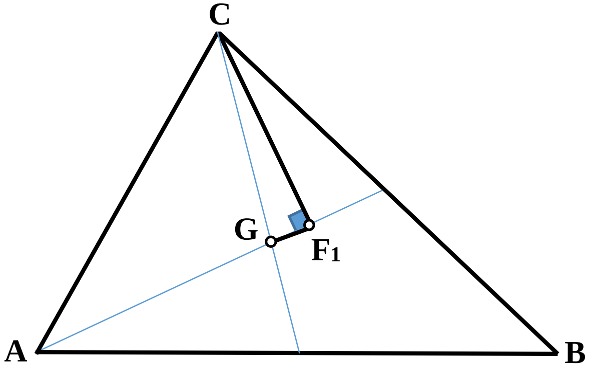

Figure 10: Example # 10

Proof:- From our earlier theorems ([1] and art 4.0 above) we know that:

$$CF_1^2 = \frac{1}{6} \cdot \frac{\left[\sqrt{3}\left(a^2 + b^2 - c^2\right) + 4\Delta\right]^2}{\left(a^2 + b^2 + c^2\right) + 4\sqrt{3}\Delta}, CG^2 = \frac{2\left(a^2 + b^2\right) - c^2}{9}, GF_1^2 = \frac{a^2 + b^2 + c^2 - 4\sqrt{3}\Delta}{18}$$

Let us first assume that: $CF_1 \perp GF_1 \Rightarrow CF_1^2 + GF_1^2 = CG^2$

$$\Rightarrow \frac{1}{6} \cdot \frac{\left[\sqrt{3}\left(a^2 + b^2 - c^2\right) + 4\Delta\right]^2}{\left(a^2 + b^2 + c^2\right) + 4\sqrt{3}\Delta} + \frac{a^2 + b^2 + c^2 - 4\sqrt{3}\Delta}{18} = \frac{2\left(a^2 + b^2\right) - c^2}{9}$$

$$\Rightarrow 5\left(a^4 + b^4 + c^4\right) + 10a^2b^2 - 8b^2c^2 - 8a^2c^2 + 12\sqrt{3}\Delta\left(a^2 + b^2 - c^2\right)$$
$$= 2a^4 + 2b^4 - c^4 + 4a^2b^2 + b^2c^2 + c^2a^2 + 4\sqrt{3}\Delta\left(2a^2 + 2b^2 - c^2\right)$$

$$\Rightarrow 3\left(a^4 + b^4 + 2c^4 + 2a^2b^2 - 3b^2c^2 - 3a^2c^2\right) + 4\sqrt{3}\Delta\left(a^2 + b^2 - 2c^2\right) = 0$$

$$\Rightarrow \sqrt{3}\left(a^2 + b^2 - c^2\right)\left(a^2 + b^2 - 2c^2\right) + 4\Delta\left(a^2 + b^2 - 2c^2\right) = 0$$

$$\Rightarrow a^2+b^2-2c^2=0 \Rightarrow \boxed{CF_1 \perp GF_1 \Rightarrow c^2=\frac{a^2+b^2}{2}}$$

Conversely we now assume that $c^2=\dfrac{a^2+b^2}{2}$ and we now consider the expression:

$$BF_1^2+GF_1^2-CG^2=\frac{1}{18}\cdot\frac{9\left(a^2+b^2-c^2\right)^2+\left(a^2+b^2+c^2\right)^2+24\sqrt{3}\Delta\left(a^2+b^2-c^2\right)}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta}-\frac{4\left(a^2+b^2\right)-2c^2}{18}$$

$$\Rightarrow BF_1^2+GF_1^2-CG^2=\frac{c^2}{3}\left[\frac{3c^2+4\sqrt{3}\Delta}{3c^2+4\sqrt{3}\Delta}\right]-\frac{c^2}{3}=0 \Rightarrow \boxed{BF_1^2+GF_1^2=CG^2}$$

$$\therefore\ c^2=\frac{a^2+b^2}{2} \Rightarrow BF_1^2+GF_1^2=CG^2 \Rightarrow CF_1 \perp GF_1 \therefore \boxed{CF_1 \perp GF_1 \Leftrightarrow c^2=\frac{a^2+b^2}{2}} \quad \text{[QED]}$$

6.0 2nd Napoleon Point, N2 (X18):

<u>Theorem</u>: Figure 11 shows the second *Napoleon* Point **N2** (**X18**). In this figure again arbitrary triangle ***ABC*** is shown and the second *Napoleon* Point is defined as the point of concurrency of the three lines: AG_1', BG_2', CG_3' - where G_1', G_2', G_3' represent the *centroids* of the inner *equilateral* triangles ***A1CB***, ***B1AC*** and ***C1BA*** respectively. With the PLAGIOgonal system chosen as shown and from our foregoing analysis we can say that the co-ordinates of the above *centroids* are given as: $\left(\dfrac{\sqrt{3}a\left(\sqrt{3}\sin B-\cos B\right)}{6\sin B}, \dfrac{\sqrt{3}a}{6\sin B}\right)$,

$$\left(\frac{\sqrt{3}\left\{a\left(\sqrt{3}\sin B+\cos B\right)-c\right\}}{6\sin B}, \frac{\sqrt{3}\left\{c\left(\sqrt{3}\cos B+\sin B\right)-a\right\}}{6\sin B}\right), \left(\frac{\sqrt{3}c}{6\sin B}, \frac{\sqrt{3}c\left(\sqrt{3}\sin B-\cos B\right)}{6\sin B}\right)$$

And consequently, the equations of the lines AG_1', BG_2', CG_3' are:

$$y=c+\frac{\left(a-2\sqrt{3}c\sin B\right)}{a\left(\sqrt{3}\sin B-\cos B\right)}x,\ y=\frac{c\left(\sqrt{3}\sin B+\cos B\right)-a}{a\left(\sqrt{3}\sin B+\cos B\right)-c}x,\ y=\frac{c\left(\sqrt{3}\sin B-\cos B\right)}{\left(c-2\sqrt{3}a\sin B\right)}x-\frac{ca\left(\sqrt{3}\sin B-\cos B\right)}{\left(c-2\sqrt{3}a\sin B\right)}$$

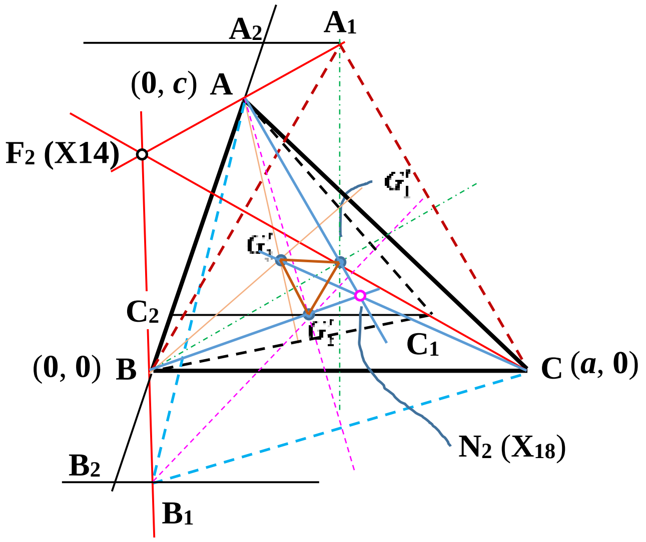

Figure 11: Second Napoleon's point (**N2** or **X18**)

Now solving the first and the last of the above three we get the point of intersection of the lines $\boldsymbol{AG_1'}$ and $\boldsymbol{BG_2'}$ as:

$$\left(\frac{2ca\sin\left(30^0-B\right)\left[2a\sin\left(B+30^0\right)-c\right]}{ca\left[4\sin^2\left(B-30^0\right)-1-12\sin^2 B\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B},\frac{2ca\sin\left(30^0-B\right)\left[2c\sin\left(B+30^0\right)-a\right]}{ca\left[4\sin^2\left(30^0-B\right)-12\sin^2 B-1\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B}\right)$$

and it is easy to see that the above point of intersection lies on the line $\boldsymbol{CG_3'}$ and hence we conclude: $\boldsymbol{AG_1', BG_2', CG_3'}$ are concurrent and thus the required co-ordinates of $\mathbf{N_2}$ or $\mathbf{X_{18}}$ are given by:

$$\left(\frac{2ca\sin\left(30^0-B\right)\left[2a\sin\left(B+30^0\right)-c\right]}{ca\left[4\sin^2\left(B-30^0\right)-1-12\sin^2 B\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B},\frac{2ca\sin\left(30^0-B\right)\left[2c\sin\left(B+30^0\right)-a\right]}{ca\left[4\sin^2\left(30^0-B\right)-12\sin^2 B-1\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B}\right)$$

and for reference, we note the co-ordinates of $\mathbf{N_1}$ or $\mathbf{X_{17}}$ are [1]:

$$\left(\frac{2ca\sin\left(30^0+B\right)\left[2a\sin\left(30^0-B\right)-c\right]}{ca\left[4\sin^2\left(B-30^0\right)-12\sin^2 B-1\right]-2\sqrt{3}\left(a^2+c^2\right)\sin B},\frac{2ca\sin\left(30^0+B\right)\left[2c\sin\left(30^0-B\right)-a\right]}{ca\left[4\sin^2\left(B-30^0\right)-12\sin^2 B-1\right]-2\sqrt{3}\left(a^2+c^2\right)\sin B}\right)$$

It is easy to observe that: co-ordinates of the *centroid* of $\boldsymbol{\Delta G_1'G_2'G_3'}$ are: $\left(\frac{a}{3},\frac{c}{3}\right)\equiv \boldsymbol{G}$ —— centroidal co-ordinates of the triangle $\boldsymbol{ABC}$. In other words, *centroid* of $\boldsymbol{\Delta G_1'G_2'G_3'}$ coincide with the centroid of the triangle $\boldsymbol{ABC}$ —— the property shared by the first Napoleon point also i.e., *centroid*s of $\boldsymbol{\Delta G_1G_2G_3}$ and $\boldsymbol{\Delta G_1'G_2'G_3'}$ are coincident.

Now we note from the triangle $\boldsymbol{\Delta BG_1'G_3'}$ that:

$$\angle G_1'BG_3' = B-60^0 \Rightarrow \left(G_1'G_3'\right)^2=\frac{1}{3}\left[a^2+c^2-2ca\cos\left(B-60^0\right)\right]$$

$\Rightarrow \left(G_1'G_3'\right)^2=\frac{1}{3}\left[c^2+a^2-\frac{c^2+a^2-b^2}{2}-2\sqrt{3}\Delta\right]=\frac{1}{6}\left(a^2+b^2+c^2-4\sqrt{3}\Delta\right)=$ constant and is always positive (by *Weitzenbock*'s inequality in triangles) and thus we conclude from the above that: $\boldsymbol{\Delta G_1'G_2'G_3'}$ is an *equilateral* triangle – a similar property of $\mathbf{N_1}$ (since $\boldsymbol{\Delta G_1G_2G_3}$ is *equilateral* [1]). Further we note that:

$$BN_2^2=\frac{4c^2a^2\sin^2\left(30^0-B\right)}{\left\{ca\left[4\sin^2\left(30^0-B\right)-12\sin^2 B-1\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B\right\}^2}\begin{bmatrix}\left\{2a\sin\left(B+30^0\right)-c\right\}^2+\left\{2c\sin\left(B+30^0\right)-a\right\}^2\\ +2\cos B\left\{2a\sin\left(B+30^0\right)-c\right\}\left\{2c\sin\left(B+30^0\right)-a\right\}\end{bmatrix}$$

$$=\frac{4c^2a^2\sin^2\left(30^0-B\right)}{\left[\sqrt{3}\left(a^2+c^2\right)\sin B-2\Delta\left(5\sin B+\sqrt{3}\cos B\right)\right]^2}\left[4\left(c^2+a^2\right)\sin^2 B+4ca\sin^2 B\cos B-4\sqrt{3}ca\sin^3 B\right]$$

$$\Rightarrow BN_2^2=\frac{16\Delta^2\sin^2\left(30^0-B\right)}{\left[\sqrt{3}\left(a^2+c^2\right)\sin B-2\Delta\left(5\sin B+\sqrt{3}\cos B\right)\right]^2}\left[\frac{3\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}{2}-2b^2\right]$$

$\Rightarrow BN_2^2=\frac{4c^2a^2\sin^2\left(30^0-B\right)}{\left[\sqrt{3}\left(a^2+c^2\right)-ca\left(5\sin B+\sqrt{3}\cos B\right)\right]^2}\left[\frac{3\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}{2}-2b^2\right]$ and similarly we can write that:

$$AN_2^2 = \frac{8b^2c^2\sin^2\left(30^0 - A\right)\cdot\left[2\Delta\sin 2A + \left(a^2+b^2+c^2\right) - 4\sqrt{3}\Delta\right]}{\left\{bc\left[4\sin^2\left(30^0 - A\right) - 12\sin^2 A - 1\right] + 2\sqrt{3}\left(b^2+c^2\right)\right\}^2} = \frac{4b^2c^2\sin^2\left(30^0 - A\right)}{\left[\sqrt{3}\left(b^2+c^2\right) - bc\left(5\sin A + \sqrt{3}\cos A\right)\right]^2}\left[\frac{3\left(a^2+b^2+c^2\right) - 4\sqrt{3}\Delta}{2} - 2a^2\right]$$

and:

$$CN_2^2 = \frac{8a^2b^2\sin^2\left(30^0 - C\right)\cdot\left[2\Delta\sin 2C + \left(a^2+b^2+c^2\right) - 4\sqrt{3}\Delta\right]}{\left\{ab\left[4\sin^2\left(30^0 - C\right) - 12\sin^2 C - 1\right] + 2\sqrt{3}\left(a^2+b^2\right)\right\}^2} = \frac{4a^2b^2\sin^2\left(30^0 - C\right)}{\left[\sqrt{3}\left(a^2+b^2\right) - ab\left(5\sin C + \sqrt{3}\cos C\right)\right]^2}\left[\frac{3\left(a^2+b^2+c^2\right) - 4\sqrt{3}\Delta}{2} - 2c^2\right]$$

In the above expressions we note:

$$2bc\sin\left(30^0 - A\right) = bc\left(\cos A - \sqrt{3}\sin A\right) = \frac{b^2+c^2-a^2}{2} - 2\sqrt{3}\Delta = \frac{a^2+b^2+c^2-4\sqrt{3}\Delta}{2} - a^2$$

$$AN_2^2 = \frac{4b^2c^2\sin^2\left(30^0 - A\right)}{\left[\sqrt{3}\left(b^2+c^2\right) - bc\left(5\sin A + \sqrt{3}\cos A\right)\right]^2}\left[b^2+c^2+2bc\sin\left(30^0 - A\right)\right]$$

$$BN_2^2 = \frac{4c^2a^2\sin^2\left(30^0 - B\right)}{\left[\sqrt{3}\left(a^2+c^2\right) - ca\left(5\sin B + \sqrt{3}\cos B\right)\right]^2}\left[c^2+a^2+2ca\sin\left(30^0 - B\right)\right]$$

$$CN_2^2 = \frac{4a^2b^2\sin^2\left(30^0 - C\right)}{\left[\sqrt{3}\left(a^2+b^2\right) - ab\left(5\sin C + \sqrt{3}\cos C\right)\right]^2}\left[a^2+b^2+2ab\sin\left(30^0 - C\right)\right]$$

And for reference we note [1]:

$$AN_1{}^2 = \frac{4b^2c^2\sin^2\left(30^0 + A\right)}{\left[\sqrt{3}\left(b^2+c^2\right) + bc\left(5\sin A - \sqrt{3}\cos A\right)\right]}\left[b^2+c^2+2bc\sin\left(30^0 + A\right)\right]$$

$$BN_1{}^2 = \frac{4c^2a^2\sin^2\left(30^0 + B\right)}{\left[\sqrt{3}\left(c^2+a^2\right) + ca\left(5\sin B - \sqrt{3}\cos B\right)\right]}\left[c^2+a^2+2ca\sin\left(30^0 + B\right)\right]$$

$$CN_1{}^2 = \frac{4a^2b^2\sin^2\left(30^0 + C\right)}{\left[\sqrt{3}\left(a^2+b^2\right) + ab\left(5\sin C - \sqrt{3}\cos C\right)\right]}\left[a^2+b^2+2ab\sin\left(30^0 + C\right)\right]$$

Finally, we note from the above co-ordinates of **N2** that for a triangle ***ABC*** with $\angle ABC = 120^0 \Rightarrow$ the co-ordinates of **N2** or **X18** are: $\left(\frac{2ca}{3\left(a-c\right)}, -\frac{2ca}{3\left(a-c\right)}\right) \Rightarrow$ **N2** lies on the *external angle bisector* of angle **B**.

7.0 Conclusion:

In the present work an extensive discussion has been attempted focussing on the second *Fermat* point and the second *Napoleon* point of a triangle using the PLAGIOgonal system developed by the first author [2], [3]. This work is considered as the extension of the earlier research work [1] focussing on the First *Fermat* and the First *Napoleon* point only. It is observed that there are lots of similarities between the various theorems and the properties of these pairs of isogonic points **X13**, **X14** and **X17** and **X18**. The discussion is further exemplified by solving various challenging problems available in the social media.

8.0 Acknowledgement:

Both the authors are deeply indebted to Sir Rachid Iksi [4] for using freely his challenging geometry problems which he authored and posted in the social media for the benefit of the mathematical community.

9.0 References:

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