ROMANIAN MATHEMATICAL MAGAZINE

If $a,b>0;a+b=2a^2b^2$ then:

$$\frac{a+1}{b^2}+\frac{b+1}{a^2}\ge 4$$

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Denote $S=a+b;P=ab;S>0;P>0$

$$a+b\overset{AM-GM}{\ge}2\sqrt{ab}\Rightarrow S\ge 2\sqrt{P}\Rightarrow 2P^2\ge 2\sqrt{P}$$

$$\Rightarrow P^2\ge P\Rightarrow P^4\ge P\Rightarrow P^3\ge 1\Rightarrow P\ge 1\Rightarrow P-1\ge 0\quad (1)$$

We used the hypothesis: $S=2P^2$.

$$\frac{a+1}{b^2}+\frac{b+1}{a^2}\ge 4\Leftrightarrow a^2(a+1)+b^2(b+1)\ge 4a^2b^2$$

$$a^3+b^3+a^2+b^2-4a^2b^2\ge 0$$

$$S^3-3SP+S^2-2P-4P^2\ge 0$$

$$8P^6-6P^3+4P^4-2P-4P^2\ge 0$$

$$4P^5-3P^2+2P^3-1-2P\ge 0$$

$$4P^5+2P^3-3P^2-2P-1\ge 0$$

$$4P^5-4P^4+4P^4-4P^3+6P^3-6P^2+3P^2-3P+P-1\ge 0$$

$$4P^4(P-1)+4P^3(P-1)+6P^2(P-1)+3P(P-1)+(P-1)\ge 0$$

$$(P-1)(4P^4+4P^3+6P^2+3P+1)\ge 0$$

$$P-1\ge 0\quad \text{(True by (1))}$$

Equality holds for $a=b=1$.