ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = 1$ and $\lambda \le \frac{9}{4}$ then :

$$\sum_{cyc} xy - \lambda xyz \le \frac{9-\lambda}{27}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} xy - \lambda xyz \overset{?}{\le} \frac{9-\lambda}{27}$$

$$\Leftrightarrow \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - \frac{1}{3}\left(\sum_{cyc} x\right)^3 + \frac{\lambda}{27}\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right) \overset{?}{\le} 0 \left(\because \sum_{cyc} x = 1\right)$$

and $\because \left(\sum_{cyc} x\right)^3 - 27xyz \overset{AM-GM}{\ge} 0$ and $\because \lambda \le \frac{9}{4} \therefore$ it suffices to prove :

$$\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - \frac{1}{3}\left(\sum_{cyc} x\right)^3 + \frac{1}{12}\left(\left(\sum_{cyc} x\right)^3 - 27xyz\right) \overset{?}{\le} 0$$

$$\Leftrightarrow \left(\sum_{cyc} x\right)^3 + 9xyz \overset{?}{\ge} 4\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \Leftrightarrow \sum_{cyc} x^3 + 3xyz \overset{?}{\ge} \sum_{cyc} x^2y + \sum_{cyc} xy^2$$

$\rightarrow$ true via Schur $\therefore \sum_{cyc} xy - \lambda xyz \le \frac{9-\lambda}{27}\ \forall x, y, z > 0 \mid x + y + z = 1$ and $\lambda \le \frac{9}{4}$,

$'' ='' $ iff $x = y = z = \frac{1}{3}$ (QED)