A FEW NON-ELEMENTARY GEOMETRICAL PROOFS

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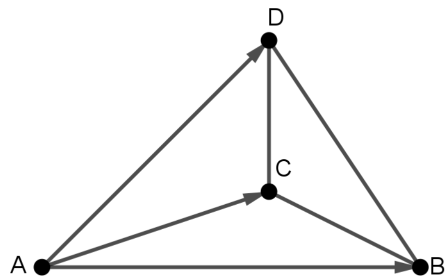
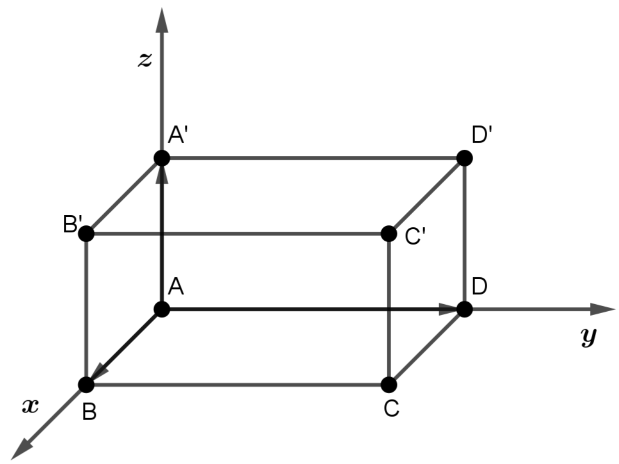
VOLUME OF THE TETRAHEDRON



Let be $A(1,1,1); B(1,2,3); C(2,3,1); D(4,4,4)$.
Find the volume of the tetrahedron $ABCD$.

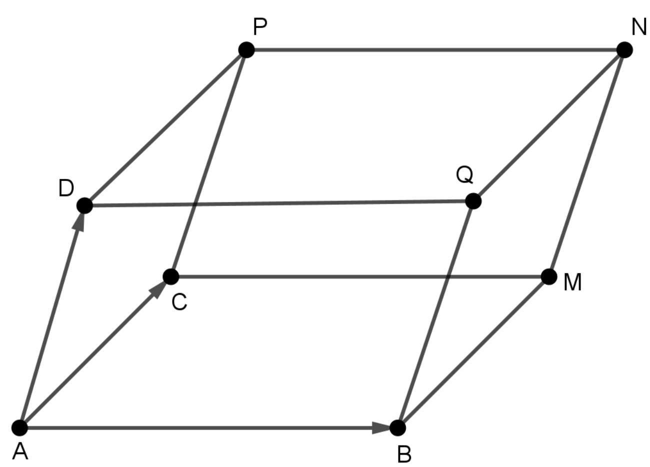
$$\overrightarrow{AB} = (x_B - x_A)\overrightarrow{i} + (y_B - y_A)\overrightarrow{j} + (z_B - z_A)\overrightarrow{k} =$$
$$= (1-1)\overrightarrow{i} + (2-1)\overrightarrow{j} + (3-1)\overrightarrow{k} = \overrightarrow{j} + 2\overrightarrow{k}$$
$$\overrightarrow{AC} = (x_C - x_A)\overrightarrow{i} + (y_C - y_A)\overrightarrow{j} + (z_C - z_A)\overrightarrow{k} =$$
$$= (2-1)\overrightarrow{i} + (3-1)\overrightarrow{j} + (1-1)\overrightarrow{k} = \overrightarrow{i} + 2\overrightarrow{j}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = 2\overrightarrow{j} - \overrightarrow{k} - 4\overrightarrow{i} = -4\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}$$
$$\overrightarrow{AD} = (x_D - x_A)\overrightarrow{i} + (y_D - y_A)\overrightarrow{j} + (z_D - z_A)\overrightarrow{k} =$$
$$= (4-1)\overrightarrow{i} + (4-1)\overrightarrow{j} + (4-1)\overrightarrow{k} = 3\overrightarrow{i} + 3\overrightarrow{j} + 3\overrightarrow{k}$$
$$\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = -4 \cdot 3 + 2 \cdot 3 - 1 \cdot 3 = -12 + 6 - 3 = -9$$
$$V[ABCD] = \frac{1}{6}|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})| = \frac{1}{6}|-9| = \frac{3}{2}$$

VOLUME OF THE RIGHT PARALLELIPIPED



$$B(L,0,0);D(0,l,0);A'(0,0,h)$$
$$\overrightarrow{AB}=L\,\overrightarrow{i};\overrightarrow{AD}=l\,\overrightarrow{j};\overrightarrow{AA'}=h\,\overrightarrow{k}$$
$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ L & 0 & 0\\ 0 & l & 0\end{vmatrix}=Ll\overrightarrow{k}$$
$$\overrightarrow{AA'}=(0-L)\,\overrightarrow{i}+(0-0)\,\overrightarrow{j}+(h-0)\,\overrightarrow{k}=-L\,\overrightarrow{i}+h\,\overrightarrow{k}$$
$$\overrightarrow{AA'}\cdot(\overrightarrow{AB}+\overrightarrow{AD})=$$
$$=(-L\,\overrightarrow{i}+h\,\overrightarrow{k})(0\cdot\,\overrightarrow{i}+0\cdot\,\overrightarrow{j}+Ll\,\overrightarrow{k})=$$
$$=-L\cdot 0+0\cdot 0+h\cdot L\cdot l=Llh$$
$$V[ABCDA'B'C'D']=Llh$$

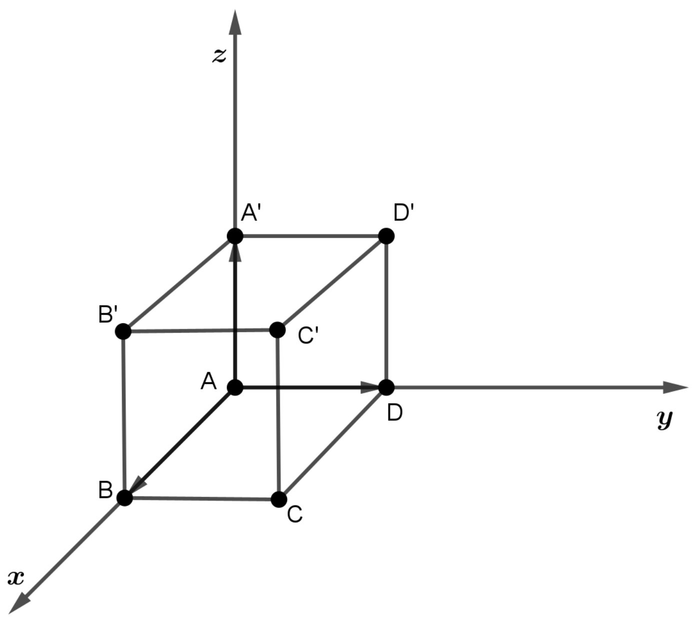
VOLUME OF THE PARALLELIPIPED



Let be $A(1,1,1); B(1,2,3); C(2,3,1); D(4,4,4)$.
Find the volume of the parallelipiped build on the vectors $\overrightarrow{AB}; \overrightarrow{AC}; \overrightarrow{AD}$.

$$\overrightarrow{AB} = (x_B - x_A)\overrightarrow{i} + (y_B - y_A)\overrightarrow{j} + (z_B - z_A)\overrightarrow{k} =$$
$$= (1-1)\overrightarrow{i} + (2-1)\overrightarrow{j} + (3-1)\overrightarrow{k} = \overrightarrow{j} + 2\overrightarrow{k}$$
$$\overrightarrow{AC} = (x_C - x_A)\overrightarrow{i} + (y_C - y_A)\overrightarrow{j} + (z_c - z_A)\overrightarrow{k} =$$
$$= (2-1)\overrightarrow{i} + (3-1)\overrightarrow{j} + (1-1)\overrightarrow{k} = \overrightarrow{i} + 2\overrightarrow{j}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{vmatrix} = 2\overrightarrow{j} - \overrightarrow{k} - 4\overrightarrow{i} = -4\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}$$
$$\overrightarrow{AD} = (x_D - x_A)\overrightarrow{i} + (y_D - y_A)\overrightarrow{j} + (z_D - z_A)\overrightarrow{k} =$$
$$= (4-1)\overrightarrow{i} + (4-1)\overrightarrow{j} + (4-1)\overrightarrow{k} = 3\overrightarrow{i} + 3\overrightarrow{j} + 3\overrightarrow{k}$$
$$\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = -4 \cdot 3 + 2 \cdot 3 - 1 \cdot 3 = -12 + 6 - = -9$$
$$V[ABMCDQNP] = |\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})| = 9$$

VOLUME OF THE CUBE



$$B(l,0,0); D(0,l,0); A'(0,0,l)$$
$$\overrightarrow{AB} = l\overrightarrow{i}; \overrightarrow{AD} = l\overrightarrow{j}; \overrightarrow{AA'} = l\overrightarrow{k}$$
$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ l & 0 & 0 \\ 0 & l & 0 \end{vmatrix} = l^2\overrightarrow{k}$$
$$\overrightarrow{AA'} = (0-l)\overrightarrow{i} + 0\overrightarrow{j} + l\overrightarrow{k}$$
$$\overrightarrow{AA'} \cdot (\overrightarrow{AB} + \overrightarrow{AD}) =$$

$$= (0\overrightarrow{i} + 0\overrightarrow{j} + l^2\overrightarrow{k}) \cdot (-l\overrightarrow{i} + 0\overrightarrow{j} + l\overrightarrow{k}) =$$
$$= 0 \cdot (-l) + 0 \cdot 0 + l^2 \cdot l = l^3$$
$$V[ABCDA'B'C'D'] = l^3$$

THE MEDIATOR PLAN OF A SEGMENT

Let be $A(1,5,1); B(2,3,4)$.
Find the mediator plan of AB.
Let M be the middle of AB.

$$M\left(\frac{1+2}{2}; \frac{5+3}{2}; \frac{1+4}{2}\right) \Rightarrow M\left(\frac{3}{2}, 4, \frac{5}{2}\right)$$
$$\overrightarrow{AB} = (x_B - x_A)\overrightarrow{i} + (y_B - y_A)\overrightarrow{j} + (z_B - z_A)\overrightarrow{k}$$
$$\overrightarrow{AB} = (2-1)\overrightarrow{i} + (3-5)\overrightarrow{j} + (4-1)\overrightarrow{j}$$
$$\overrightarrow{AB} = \overrightarrow{i} - 2\overrightarrow{j} + 3\overrightarrow{j}$$

Let P be the mediator plan of AB.

$$P : (x - x_M) \cdot 1 + (y - y_M) \cdot (-2) + (z - z_M) \cdot 3 = 0$$
$$P : \left(x - \frac{3}{2}\right) + (y-4) \cdot (-2) + \left(z - \frac{5}{2}\right) \cdot 3 = 0$$
$$P : x - \frac{3}{2} - 2y + 8 + 3z - \frac{15}{2} = 0$$
$$P : x - 2y + 3z - 1 = 0$$

THE DISTANCE FROM A POINT TO A REAL PLAN

1. Let be $A(2,1,3)$ and the real plan:

$$P : 3x + 4y + 5z - 10 = 0$$

Find the distance from A to P.

$$d(A,P) = \frac{|3x_A + 4y_A + 5z_A - 10|}{\sqrt{3^2 + 4^2 + 5^2}}$$
$$d(A,P) = \frac{|3 \cdot 2 + 4 \cdot 1 + 5 \cdot 3 - 10|}{\sqrt{9 + 16 + 25}} = \frac{|6 + 4 + 15 - 10|}{\sqrt{50}}$$
$$d(A,P) = \frac{15}{5\sqrt{2}} = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$$

2. Let be $A(1,1,0); B(0,1,0); C(0,1,1); D(8,8,8)$.
Find the distance from D to the real plan (ABC).

$$(ABC) : \begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$(ABC) : \begin{vmatrix} x & y-1 & z-1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} = 0$$

$$(ABC) : \begin{vmatrix} x & y-1 & z-1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{vmatrix} = 0$$

$$(ABC) : y - 1 = 0$$

$$d(D, (ABC)) = \frac{|8-1|}{\sqrt{1^2+0^2+0^2}} = 7$$

THE DISTANCE FROM A POINT TO A LINE

Let be $A(2,-1,1); B(0,1,3); C(-1,2,2)$.
Find the distance from the point A to the line BC.

$$\overrightarrow{AB} = (x_B - x_A)\overrightarrow{i} + (y_b - y_A)\overrightarrow{j} + (z_B - z_A)\overrightarrow{k} =$$

$$= (0-2)\overrightarrow{i} + (1+1)\overrightarrow{j} + (3-1)\overrightarrow{k} = -2\overrightarrow{i} + 2\overrightarrow{j} + 2\overrightarrow{k}$$

$$\overrightarrow{AC} = (x_C - x_A)\overrightarrow{i} + (y_C - y_A)\overrightarrow{j} + (z_C - z_A)\overrightarrow{k}$$

$$= (-1-2)\overrightarrow{i} + (2+1)\overrightarrow{j} + (2-1)\overrightarrow{k} = -3\overrightarrow{i} + 3\overrightarrow{j} + \overrightarrow{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -2 & 2 & 2 \\ -3 & 3 & 1 \end{vmatrix} = 2\overrightarrow{i} - 6\overrightarrow{j} - 6\overrightarrow{k} + 6\overrightarrow{k} - 6\overrightarrow{i} + 2\overrightarrow{j}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -4\overrightarrow{i} - 4\overrightarrow{j}$$

$$A[ABC] = \frac{1}{2}\sqrt{(-4)^2 + (-4)^2} = \frac{32}{2} = 2\sqrt{2}$$

$$BC = \sqrt{(x_C - x_B)^2 + (y_C - y_B)^2 + (z_C - z_B)^2} =$$

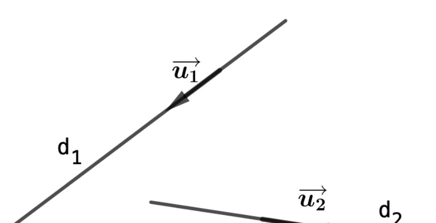
$$= \sqrt{(-1-0)^2 + (2-1)^2 + (2-3)^2} = \sqrt{3}$$

$$d(A, BC) = \frac{2A[ABC]}{BC} = \frac{2 \cdot 2\sqrt{2}}{\sqrt{3}} = \frac{4\sqrt{2}}{\sqrt{3}} = \frac{4\sqrt{6}}{3}$$

THE ANGLE BETWEEN TWO LINES

Let be the lines:

$$d_1 : \begin{cases} x = 2 + t \\ y = 3 + 2t \\ z = 1 + 3t \end{cases} \quad ; \qquad d_2 : \begin{cases} x = 3 + t \\ y = 1 + 3t \\ z = 4 + 2t \end{cases}$$



$$d_1 : \begin{cases} x - 2 = t \\ \frac{y-3}{2} = t \\ \frac{z-1}{3} = t \end{cases} \quad ; \qquad d_2 : \begin{cases} x - 3 = t \\ \frac{y-1}{3} = t \\ \frac{z-4}{2} - t \end{cases} \quad ; \qquad t \in \mathbb{R}$$

$$d_1 : \frac{x-2}{1} = \frac{y-3}{2} = \frac{z-1}{3}; d_2 : \frac{x-3}{1} = \frac{y-1}{3} = \frac{z-4}{2}$$

$$\overrightarrow{u_1}(1,2,3) = \overrightarrow{i} + 2\overrightarrow{j} + 3\overrightarrow{k}$$

$$u_2(1,3,2) = \overrightarrow{i} + 3\overrightarrow{j} + 2\overrightarrow{k}$$

$$\overrightarrow{u_1} \cdot \overrightarrow{u_2} = 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 2 = 13$$

$$|\overrightarrow{u_1}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$|\overrightarrow{u_2}| = \sqrt{1^2 + 3^2 + 2^2} = \sqrt{14}$$

$$\cos(\angle(\overrightarrow{u_1}, \overrightarrow{u_2})) = \frac{\overrightarrow{u_1} \cdot \overrightarrow{u_2}}{|\overrightarrow{u_1}| \cdot |\overrightarrow{u_2}|} = \frac{13}{\sqrt{14} \cdot \sqrt{14}} = \frac{13}{14}$$

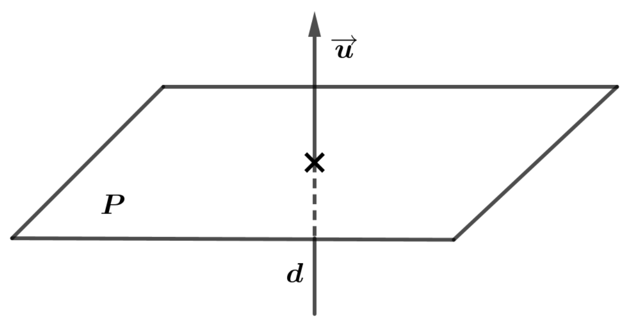
$$\mu(\angle(\overrightarrow{u_1}, \overrightarrow{u_2})) = \arccos\left(\frac{13}{14}\right)$$

PERPENDICULAR LINE ON THE REAL PLAN

Let be $A(1,1,1)$ and the real plan:

$$P : x + 2y + 3z - 4 = 0$$

Find the equations of the perpendicular line from A to the real plan P.



$$\overrightarrow{u}(1,2,3)$$

$$\overrightarrow{u} = \overrightarrow{i} + 2\overrightarrow{j} + 3\overrightarrow{k}$$

$$d : \frac{x - x_A}{1} = \frac{y - y_A}{2} = \frac{z - z_A}{3}$$

$$d : x - 1 = \frac{y-1}{2} = \frac{z-1}{3}$$

Let's find also the parametrical equations of d:

$$x - 1 = t; \frac{y-1}{2} = t; \frac{z-1}{3} = t$$

$$d : \begin{cases} x = 1 + t \\ y = 1 + 2t \\ z = 1 + 3t \end{cases} ; t \in \mathbb{R}$$

AREA OF THE TRIANGLE

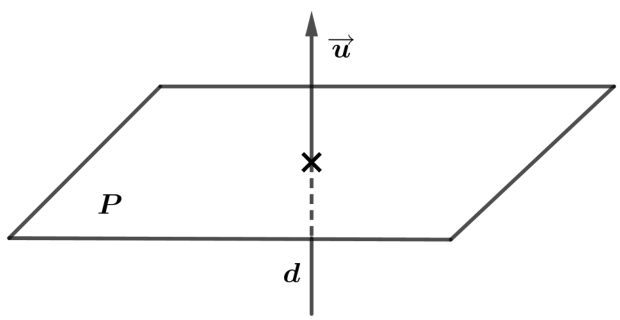
Let be $A(1,1,0); B(0,1,1); C(2,2,2)$.
Find the area of ΔABC.

$$\overrightarrow{AB} = (x_B - x_A)\overrightarrow{i} + (y_B - y_A)\overrightarrow{j} + (z_B - z_A)\overrightarrow{k}$$
$$\overrightarrow{AB} = (0-1)\overrightarrow{i} + (1-1)\overrightarrow{j} + (1-0)\overrightarrow{k} = -\overrightarrow{i} + \overrightarrow{k}$$
$$\overrightarrow{AC} = (x_C - x_A)\overrightarrow{i} + (y_C - y_A)\overrightarrow{j} + (z_C - z_A)\overrightarrow{k}$$
$$\overrightarrow{AC} = (2-1)\overrightarrow{i} + (2-1)\overrightarrow{j} + (2-0)\overrightarrow{k} = \overrightarrow{i} + \overrightarrow{j} + 2\overrightarrow{k}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \overrightarrow{j} - \overrightarrow{k} - \overrightarrow{i} + 2\overrightarrow{j}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = -\overrightarrow{i} + 3\overrightarrow{j} - \overrightarrow{k}$$
$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-1)^2 + 3^2 + (-1)^2} = \sqrt{11}$$
$$\text{Area } (\Delta ABC) = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{\sqrt{11}}{2}$$

Let be $A(1,1,1)$ and the real plan:

$$P : x + 2y + 3z - 4 = 0$$

Find the equations of the perpendicular line from A to the real plan P.



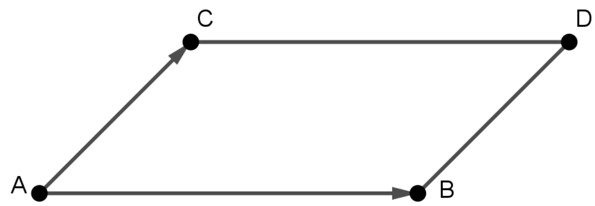
$$\overrightarrow{u}(1,2,3)$$
$$\overrightarrow{u} = \overrightarrow{i} + 2\overrightarrow{j} + 3\overrightarrow{k}$$
$$d : \frac{x - x_A}{1} = \frac{y - y_A}{2} = \frac{z - z_A}{3}$$
$$d : x - 1 = \frac{y-1}{2} = \frac{z-1}{3}$$

Let's find also the parametrical equations of d:

$$x - 1 = t; \frac{y-1}{2} = t; \frac{z-1}{3} = t$$
$$d : \begin{cases} x = 1 + t \\ y = 1 + 2t \qquad ; t \in \mathbb{R} \\ z = 1 + 3t \end{cases}$$

THE AREA OF THE PARALLELOGRAM

Let be $A(2,0,0); B(0,1,0); C(1,2,2)$.
Find the area of the parallelogram builded on $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



$$\overrightarrow{AB} = (x_B - x_A)\overrightarrow{i} + (y_B - y_A)\overrightarrow{j} + (z_B - z_A)\overrightarrow{k}$$

$$\overrightarrow{AB} = (0-2)\overrightarrow{i} + (1-0)\overrightarrow{j} + (0-0)\overrightarrow{k} = -2\overrightarrow{i} + \overrightarrow{j}$$

$$\overrightarrow{AC} = (x_C - x_A)\overrightarrow{i} + (y_c - y_A)\overrightarrow{j} + (z_C - z_A)\overrightarrow{k}$$

$$\overrightarrow{AC} = (1-2)\overrightarrow{i} + (2-0)\overrightarrow{j} + (2-0)\overrightarrow{k} = -\overrightarrow{i} + 2\overrightarrow{j} + 2\overrightarrow{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -2 & 1 & 0 \\ -1 & 2 & 2 \end{vmatrix} =$$

$$= 2\overrightarrow{i} - 4\overrightarrow{k} + \overrightarrow{k} + 4\overrightarrow{j} = 2\overrightarrow{i} + 4\overrightarrow{j} - 3\overrightarrow{k}$$

$$\text{Area } (ABCD) = |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{2^2 + 4^2 + (-3)^2} = \sqrt{29}$$

THE ANGLE BETWEEN TWO REAL PLANS

Let be the real plans:

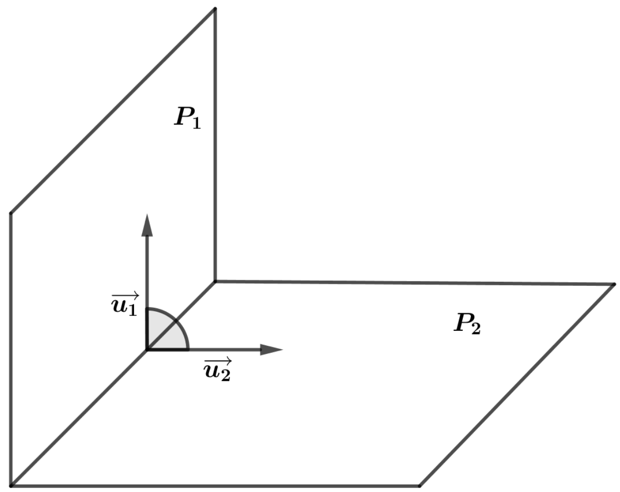
$$P_1 : x + 2y - 3z + 1 = 0$$

$$P_2 : 5x - 3y + 4z - 2 = 0$$

The normal vectors of P_1, P_2 are:

$$\overrightarrow{u_1}(1,2,-3) = \overrightarrow{i} + 2\overrightarrow{j} - 3\overrightarrow{k}$$

$$\overrightarrow{u_2}(5,-3,4) = 5\overrightarrow{i} - 3\overrightarrow{j} + 4\overrightarrow{k}$$



$$\overrightarrow{u_1} \cdot \overrightarrow{u_2} = 1 \cdot 5 + 2 \cdot (-3) - 3 \cdot 4 = 5 - 6 - 12 = -13$$

$$|\overrightarrow{u_1}| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$|\overrightarrow{u_2}| = \sqrt{5^2 + (-3)^2 + 4^2} = \sqrt{25 + 9 + 16} = \sqrt{50} = 5\sqrt{2}$$

$$\cos(\angle(P_1, P_2)) = \cos(\angle(\overrightarrow{u_1}, \overrightarrow{u_2})) =$$

$$= \frac{\overrightarrow{u_1} \cdot \overrightarrow{u_2}}{|\overrightarrow{u_1}| \cdot |\overrightarrow{u_2}|} = \frac{-13}{\sqrt{14} \cdot 5\sqrt{2}} = \frac{-13}{10\sqrt{7}} = \frac{-13\sqrt{7}}{70}$$

THE ANGLE BETWEEN A LINE AND A REAL PLAN

Let be the line:

$$d : \begin{cases} x = 1 + t \\ y = 2 + 3t \\ z = -2 + 5t \end{cases} \quad ; t \in \mathbb{R}$$

Let be the real plan:

$$P : 2x + 3y + z - 4 = 0$$

Find the angle between d and P.

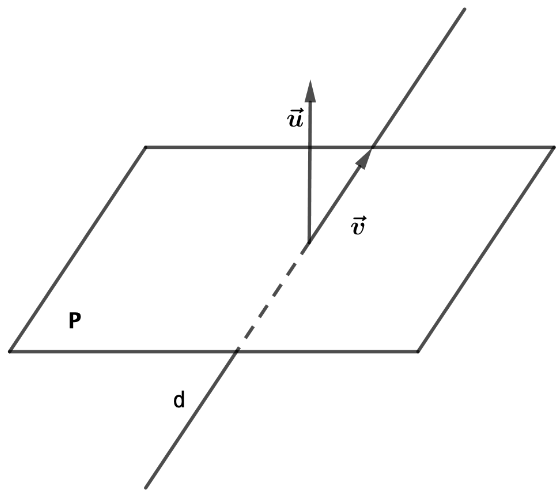
$$d : \begin{cases} x - 1 = t \\ \frac{y-2}{3} = t \\ \frac{z+2}{5} = t \end{cases} \quad ; d : \frac{x-1}{1} = \frac{y-2}{3} = \frac{z+2}{5}$$

The line d has the directory vector:

$$\overrightarrow{v}(1, 3, 5) = \overrightarrow{i} + 3\overrightarrow{j} + 5\overrightarrow{k}$$

The normal vector of the real plan P is:

$$\overrightarrow{u}(2, 3, 1) = 2\overrightarrow{i} + 3\overrightarrow{j} + \overrightarrow{k}$$



$$\overrightarrow{u}\cdot\overrightarrow{v}=1\cdot 2+3\cdot 3+5\cdot 1=16$$
$$|\overrightarrow{u}|=\sqrt{2^2+3^2+1^2}=\sqrt{14}$$
$$|\overrightarrow{v}|=\sqrt{1^2+3^2+5^2}=\sqrt{35}$$
$$\cos(\angle(\overrightarrow{u},\overrightarrow{v}))=\frac{\overrightarrow{u}\cdot\overrightarrow{v}}{|\overrightarrow{u}|\cdot|\overrightarrow{v}|}=\frac{16}{\sqrt{14}\cdot\sqrt{15}}=\frac{16}{\sqrt{210}}=\frac{8\sqrt{210}}{105}$$

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