

5834

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5834. Evaluate

$$\Omega = \int_0^\pi \frac{(x+2)\sin x}{1+3\cos^2 x} dx$$

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Solution 1 by Zuming Bi, Academy of Mathematics and Science, Hays, KS; and Paul Flesher, Fort Hays State University, Hays, KS; and Jonathan Rehmert, Fort. Hays State University, Hays, KS

Method 1: Partial Fraction Decomposition

$$\begin{aligned}\Omega &= \int_0^\pi \frac{(x+2)\sin x}{1+3\cos^2 x} dx = \int_0^\pi \frac{(x+2)\sin x}{4-3\sin^2 x} dx \\ &= \int_0^\pi \frac{(x+2)\sin x}{(2+\sqrt{3}\sin x)(2-\sqrt{3}\sin x)} dx\end{aligned}$$

which we re-express as

$$\begin{aligned}\Omega &= \int_0^\pi \frac{1}{2\sqrt{3}} \left(\frac{-(x+2)(2-\sqrt{3}\sin x) + (x+2)(2+\sqrt{3}\sin x)}{4-3\sin^2 x} \right) dx \\ &= \int_0^\pi -\frac{(x+2)(2-\sqrt{3}\sin x)}{2\sqrt{3}(4-3\sin^2 x)} + \frac{(x+2)(2+\sqrt{3}\sin x)}{2\sqrt{3}(4-3\sin^2 x)} dx \\ &= \int_0^\pi -\frac{x+2}{2\sqrt{3}(2+\sqrt{3}\sin x)} + \frac{x+2}{2\sqrt{3}(4-3\sin x)} dx \\ &= \int_0^\pi -\frac{x+2}{4\sqrt{3}+6\sin x} + \frac{x+2}{4\sqrt{3}-6\sin x} dx\end{aligned}$$

Look at the latter separately, we have

$$\int_0^\pi \frac{x+2}{4\sqrt{3}-6\sin x} dx = \int_0^\pi \frac{x}{4\sqrt{3}-6\sin x} dx + \int_0^\pi \frac{2}{4\sqrt{3}-6\sin x} dx.$$

Let

$$\begin{aligned}k &= \int_0^\pi \frac{x}{4\sqrt{3}-6\sin x} dx \\ p &= \int_0^\pi \frac{2}{4\sqrt{3}-6\sin x} dx.\end{aligned}$$

Do a u substitution:

$$\begin{aligned}u &= \pi - x \\ k &= \int_\pi^0 \frac{\pi - u}{4\sqrt{3}-6\sin(\pi - u)} (-du) = \int_0^\pi \frac{\pi - u}{4\sqrt{3}-6\sin(u)} du\end{aligned}$$

Note the trig identity

$$\sin(\pi - u) = \sin(u).$$

Change to equivalent integral

$$\begin{aligned}
 k &= \int_0^\pi \frac{\pi - u}{4\sqrt{3} - 6\sin(u)} du = \int_0^\pi \frac{\pi - x}{4\sqrt{3} - 6\sin(x)} dx \\
 2k &= \int_0^\pi \frac{\pi - x}{4\sqrt{3} - 6\sin(x)} dx + \int_0^\pi \frac{x}{4\sqrt{3} - 6\sin x} dx = \int_0^\pi \frac{\pi}{4\sqrt{3} - 6\sin x} dx \\
 2k &= \pi \int_0^\pi \frac{1}{4\sqrt{3} - 6\sin x} dx \\
 k &= \frac{\pi}{2} \int_0^\pi \frac{1}{4\sqrt{3} - 6\sin x} dx = \frac{\pi}{2} \times \frac{p}{2} = \frac{\pi}{4} p \\
 k &= \frac{\pi}{4} p.
 \end{aligned}$$

So

$$\int_0^\pi \frac{x+2}{4\sqrt{3} - 6\sin x} dx = \frac{\pi}{4} p + p = \frac{\pi+4}{4} p.$$

Let

$$\begin{aligned}
 m &= \int_0^\pi \frac{x}{4\sqrt{3} + 6\sin x} dx \\
 n &= \int_0^\pi \frac{2}{4\sqrt{3} + 6\sin x} dx.
 \end{aligned}$$

Do a u substitution

$$\begin{aligned}
 u &= \pi - x \\
 k &= \int_\pi^0 \frac{\pi - u}{4\sqrt{3} + 6\sin(\pi - u)} (-du) = \int_0^\pi \frac{\pi - u}{4\sqrt{3} + 6\sin(u)} du.
 \end{aligned}$$

Change to equivalent integral

$$\begin{aligned}
 m &= \int_0^\pi \frac{\pi - u}{4\sqrt{3} + 6\sin(u)} du = \int_0^\pi \frac{\pi - x}{4\sqrt{3} + 6\sin(x)} dx \\
 2m &= \int_0^\pi \frac{\pi - x}{4\sqrt{3} + 6\sin(x)} dx + \int_0^\pi \frac{x}{4\sqrt{3} + 6\sin x} dx = \int_0^\pi \frac{\pi}{4\sqrt{3} + 6\sin x} dx \\
 2m &= \pi \int_0^\pi \frac{1}{4\sqrt{3} + 6\sin x} dx \\
 m &= \frac{\pi}{2} \int_0^\pi \frac{1}{4\sqrt{3} + 6\sin x} dx = \frac{\pi}{2} \times \frac{n}{2} = \frac{\pi}{4} n \\
 m &= \frac{\pi}{4} n.
 \end{aligned}$$

Then

$$\int_0^\pi \frac{x+2}{4\sqrt{3} + 6\sin x} dx = \frac{\pi+4}{4} \int_0^\pi \frac{2}{4\sqrt{3} + 6\sin x} dx.$$

Deal with the other term in (1):

$$\begin{aligned}
 \Omega &= \frac{\pi+4}{4} \left(\int_0^\pi \frac{2}{4\sqrt{3} - 6\sin x} dx - \int_0^\pi \frac{2}{4\sqrt{3} + 6\sin x} dx \right) \\
 &= \frac{\pi+4}{4} \left(\int_0^\pi \frac{1}{4\sqrt{3} - 6\sin x} dx - \int_0^\pi \frac{1}{4\sqrt{3} + 6\sin x} dx \right)
 \end{aligned}$$

We have

$$\int_0^\pi \frac{1}{4\sqrt{3} - 6\sin x} dx - \int_0^\pi \frac{1}{4\sqrt{3} + 6\sin x} dx = \int_0^\pi \frac{12\sin x}{48 - 36\sin^2 x} dx.$$

Let

$$u = \cos x$$

Then

$$\begin{aligned} \int_0^\pi \frac{12\sin x}{48 - 36\sin^2 x} dx &= \int_0^\pi \frac{\sin x}{4 - 3\sin^2 x} dx = \int_1^{-1} \frac{-du}{4 - 3(1 - u^2)} = \int_{-1}^1 \frac{1}{1 + 3u^2} du \\ &= \frac{1}{\sqrt{3}} \arctan(\sqrt{3}u) \Big|_{-1}^1 = \frac{1}{\sqrt{3}} (\arctan(\sqrt{3}) - \arctan(-\sqrt{3})) = \frac{1}{\sqrt{3}} \left(\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right) \\ &= \frac{2\pi}{3\sqrt{3}} \end{aligned}$$

Then

$$\Omega = \frac{\pi + 4}{2} \times \frac{2\pi}{3\sqrt{3}} = \frac{\pi(\pi + 4)}{3\sqrt{3}}.$$

Method 2: Direct u Substitution

Let

$$u = \frac{\pi}{2} - x$$

Then

$$\Omega = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - u + 2\right) \sin\left(\frac{\pi}{2} - u\right)}{1 + 3\cos^2\left(\frac{\pi}{2} - u\right)} (-du) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - u + 2\right) \sin\left(\frac{\pi}{2} - u\right)}{1 + 3\cos^2\left(\frac{\pi}{2} - u\right)} du.$$

Note

$$\sin\left(\frac{\pi}{2} - u\right) = \cos(u).$$

$$\Omega = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - u + 2\right) \cos u}{1 + 3\sin^2 u} du = \left(\frac{\pi}{2} + 2\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos u}{1 + 3\sin^2 u} du - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{u \cos u}{1 + 3\sin^2 u} du.$$

Notice since $\cos u$ is an even function, then $\frac{\cos u}{1 + 3\sin^2 u}$ is an even function and $\frac{u \cos u}{1 + 3\sin^2 u}$ is an odd function. Hence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{u \cos u}{1 + 3\sin^2 u} du = 0.$$

Therefore

$$\begin{aligned} \Omega &= \left(\frac{\pi}{2} + 2\right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos u}{1 + 3\sin^2 u} du = \left(\frac{\pi}{2} + 2\right) \frac{1}{\sqrt{3}} \arctan(\sqrt{3} \sin u) \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \left(\frac{\pi}{2} + 2\right) \frac{2\pi}{3\sqrt{3}} = \frac{\pi(\pi + 4)}{3\sqrt{3}}. \end{aligned}$$

Method 3: Integration by Parts

Let

$$u = x + 2, du = dx.$$

So

$$dv = \frac{\sin x}{1 + 3\cos^2 x} dx, v = -\frac{1}{\sqrt{3}} \arctan(\sqrt{3} \cos x).$$

Then

$$\Omega = -(x + 2) \frac{1}{\sqrt{3}} \arctan(\sqrt{3} \cos x) \Big|_0^\pi + \frac{1}{\sqrt{3}} \int_0^\pi \arctan(\sqrt{3} \cos x) dx.$$

Calculate the second part:

$$u = \frac{\pi}{2} - x,$$

$$du = -dx.$$

Note

$$\cos\left(\frac{\pi}{2} - u\right) = \sin(u).$$

$$\frac{1}{\sqrt{3}} \int_0^\pi \arctan(\sqrt{3} \cos x) dx = \frac{1}{\sqrt{3}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \arctan\left(\sqrt{3} \cos\left(\frac{\pi}{2} - u\right)\right) (-du) = -\frac{1}{\sqrt{3}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \arctan(\sqrt{3} \sin u) du.$$

Since

$$f(x) = \arctan(\sqrt{3} \sin u), f(-x) = -f(x)$$

$$f(x) \text{ is an odd function, then } -\frac{1}{\sqrt{3}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \arctan(\sqrt{3} \sin u) du = 0,$$

Then

$$\begin{aligned} \Omega &= -(x+2) \frac{1}{\sqrt{3}} \arctan(\sqrt{3} \cos x) \Big|_0^\pi = (\pi+2) \left(-\frac{1}{\sqrt{3}} \arctan(-\sqrt{3}) \right) - 2 \left(-\frac{1}{\sqrt{3}} \arctan(\sqrt{3}) \right) \\ &= \frac{\pi+2}{\sqrt{3}} \times \frac{\pi}{3} + \frac{2}{\sqrt{3}} \times \frac{\pi}{3} = \frac{\pi(\pi+4)}{3\sqrt{3}} \end{aligned}$$

In conclusion

$$\Omega = \frac{\pi(\pi+4)}{3\sqrt{3}}.$$

□

Solution 2 by Prakash Pant, The University of Vermont, Bariya, Nepal.

Use the substitution $x \rightarrow \pi - x$,

$$\Omega = \int_0^\pi \frac{(x+2) \sin(x)}{1+3 \cos^2(x)} dx = \int_0^\pi \frac{(\pi-x+2) \sin(\pi-x)}{1+3 \cos^2(\pi-x)} dx = \int_0^\pi \frac{(\pi-x+2) \sin(x)}{1+3 \cos^2(x)} dx$$

Adding the first and third expression expressions,

$$2\Omega = \int_0^\pi \frac{(\pi+4) \sin(x)}{1+3 \cos^2(x)} dx$$

Use the substitution $\cos(x) \rightarrow x$,

$$2\Omega \int_1^{-1} \frac{(\pi+4)(-dx)}{1+3x^2} = \int_{-1}^1 \frac{(\pi+4)(dx)}{1+3x^2}$$

Since the integrand is even,

$$\Omega = \frac{\pi+4}{3} \int_0^1 \frac{1}{x^2 + \frac{1}{3}} = \frac{\pi+4}{3} \sqrt{3} \arctan(\sqrt{3}x) \Big|_0^1 = \frac{\pi(\pi+4)\sqrt{3}}{9}$$

which solves the given problem.

□

Solution by Péter Fulop, Gyomro, Hungary.

Integration by parts:

$$\begin{aligned} u &= x+2 & v' &= \frac{\sin(x)}{1+\cos^2(x)} \\ u' &= 1 & v &= \int \frac{\sin(x)}{1+\cos^2(x)} dx \end{aligned}$$

Calculation of $v(x)$:

Substitution: $t = \cos(x)$

$$x = \cos^{-1}(t) \quad \frac{dx}{dt} = -\frac{1}{\sqrt{1-t^2}}$$

$$v(x) = \int -\frac{1}{\sqrt{1-t^2}} \frac{\sqrt{1-t^2}}{\sqrt{1-3t^2}} dt = -\frac{\tan^{-1}(\sqrt{3}t)}{\sqrt{3}} \Big|_{t=\cos(x)}$$

Calculation of Ω :

$$\Omega_1 = 2 \frac{\tan^{-1}(\sqrt{3})}{\sqrt{3}} - (\pi + 2) \frac{\tan^{-1}(-\sqrt{3})}{\sqrt{3}} = (\pi + 4) \frac{\tan^{-1}(\sqrt{3})}{\sqrt{3}} = \frac{\pi\sqrt{3}(\pi + 4)}{9}.$$

Substitution at the second term: $z = \sqrt{3}\cos(x)$:

$$x = \cos^{-1}\left(\frac{z}{\sqrt{3}}\right) \quad \frac{dx}{dz} = -\frac{\frac{1}{\sqrt{3}}}{\sqrt{1-\frac{z^2}{3}}}$$

$$\Omega_2 = \int_0^\pi \frac{\tan^{-1}(\sqrt{3}\cos(x))}{\sqrt{3}} dx = \frac{1}{3} \int_{-\sqrt{3}}^{+\sqrt{3}} \frac{\tan^{-1}(z)}{\sqrt{1-\frac{z^2}{3}}} dz = 0$$

As the function to be integrated is odd, and its integral in a symmetric domain is zero. So $\Omega = \Omega_1$.

$$\Omega = \frac{\pi(\pi + 4)}{3\sqrt{3}}.$$

□

Solution 4 by Albert Stadler, Herrliberg, Switzerland.

Apply the change of variable $x \rightarrow \pi - x$. Since $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$, we obtain

$$\Omega = \int_0^\pi \frac{(\pi - x + 2) \sin x}{1 + 3 \cos^2 x} dx$$

Adding the original and transformed expressions yields

$$2\Omega = \int_0^\pi \frac{(\pi + 4) \sin x}{1 + 3 \cos^2 x} dx$$

Since the integrand is symmetric about $\frac{\pi}{2}$, we write

$$\int_0^\pi \frac{(\pi + 4) \sin x}{1 + 3 \cos^2 x} dx = 2 \int_0^{\frac{\pi}{2}} \frac{(\pi + 4) \sin x}{1 + 3 \cos^2 x} dx$$

Hence,

$$2\Omega = 2 \int_0^{\frac{\pi}{2}} \frac{(\pi + 4) \sin x}{1 + 3 \cos^2 x} dx$$

Let $y = \cos x$. Then $dy = -\sin x dx$, and as x goes from 0 to $\frac{\pi}{2}$, y goes from 1 to 0. Therefore,

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + 3 \cos^2 x} dx = \int_0^1 \frac{1}{1 + 3y^2} dy = \frac{1}{\sqrt{3}} \arctan(\sqrt{3}y) \Big|_0^1 = \frac{1}{\sqrt{3}} \arctan(\sqrt{3}) = \frac{\pi}{3\sqrt{3}}.$$

Thus,

$$\Omega = \frac{(\pi + 4)\pi}{3\sqrt{3}}$$

□

Solution 5 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Using the identity $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we assert

$$\Omega = \int_0^\pi \frac{(\pi - x + 2) \sin(\pi - x)}{1 + 3 \cos^2(\pi - x)} dx$$

Since $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$, then

$$\Omega = \int_0^\pi \frac{(\pi - x + 2) \sin x}{1 + 3 \cos^2 x} dx$$

So

$$\begin{aligned} 2\Omega &= \int_0^\pi \frac{(x + 2 + \pi - x + 2) \sin x}{1 + 3 \cos^2 x} dx \\ &= \int_0^\pi \frac{(\pi + 4) \sin x}{1 + 3 \cos^2 x} dx \\ &= -\frac{\pi + 4}{\sqrt{3}} \int_0^\pi \frac{-\sqrt{3} \sin x}{1 + (\sqrt{3} \cos x)^2} dx \\ &= \frac{\pi + 4}{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dt}{1 + t^2} \\ &= \frac{\pi + 4}{\sqrt{3}} [\arctan t]_{-\sqrt{3}}^{\sqrt{3}} \\ &= \frac{\pi + 4}{\sqrt{3}} \left(\frac{\pi}{3} - \left(-\frac{\pi}{3} \right) \right) \\ &= \frac{2(\pi + 4)\pi}{3\sqrt{3}} \end{aligned}$$

In short,

$$\Omega = \frac{\pi^2 + 4\pi}{3\sqrt{3}} \sim 4.3178 \dots$$

□

Solution 6 by Brian D. Beasley, Simpsonville, SC.

Using the substitution $w = \cos x$, we note that

$$\int \frac{\sin x}{1 + 3 \cos^2 x} dx = - \int \frac{dw}{1 + 3w^2} = f(x) + C,$$

where $f(x) = (-1/\sqrt{3}) \tan^{-1}(\sqrt{3} \cos x)$. This result allows us in turn to apply integration by parts with $u = x + 2, du = dx, v = f(x)$, and $dv = \sin x dx / (1 + 3 \cos^2 x)$ to yield

$$\Omega = uv \Big|_0^\pi - \int_0^\pi v du = (x + 2)f(x) \Big|_0^\pi - \int_0^\pi f(x) dx$$

We claim that the definite integral in this expression equal zero, since $f(x/2 - x) = -f(x/2 + x)$:

$$f\left(\frac{\pi}{2} - x\right) = -\frac{1}{\sqrt{3}} \tan^{-1} \left(\sqrt{3} \cos\left(\frac{\pi}{2} - x\right) \right) = -\frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3} \sin x)$$

and

$$f\left(\frac{\pi}{2}+x\right)=-\frac{1}{\sqrt{3}}\tan^{-1}\left(\sqrt{3}\cos\left(\frac{\pi}{2}+x\right)\right)=-\frac{1}{\sqrt{3}}\tan^{-1}(-\sqrt{3}\sin x)=\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\sin x)$$

In particular, we obtain

$$\int_0^{\frac{\pi}{2}} f(x)dx = -\int_{\frac{\pi}{2}}^0 f\left(\frac{\pi}{2}-u\right)du = \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-u\right)du$$

and

$$\int_{\frac{\pi}{2}}^{\pi} f(x)dx = \int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}+u\right)du = -\int_0^{\frac{\pi}{2}} f\left(\frac{\pi}{2}-u\right)du$$

Hence $\int_0^{\pi} f(x)dx = 0$, so we conclude

$$\Omega = (\pi + 2)f(\pi) - 2f(0) = -\frac{(\pi + 2)}{\sqrt{3}}\left(-\frac{\pi}{3}\right) + \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi(\pi + 4)}{3\sqrt{3}}.$$

□

Solution 7 by Devis Alvarado, UNAH and UPNFM, Tegucigalpa, Honduras.

$$\begin{aligned}\Omega &= \int_0^{\pi} \frac{(x+2)\sin(x)}{1+3\cos^2(x)}dx \\ &= \int_0^{\pi} \frac{(\pi-y+2)\sin(\pi-y)}{1+3\cos^2(\pi-y)}dy, y = \pi - x \\ &= \int_0^{\pi} \frac{(\pi+4-2-y)\sin(y)}{1+3\cos^2(y)}dy \\ &= (\pi+4) \int_0^{\pi} \frac{\sin(y)}{1+3\cos^2(y)}dy - \int_0^{\pi} \frac{(2+y)\sin(y)}{1+3\cos^2(y)}dy \\ &= (\pi+4) \int_0^{\pi} \frac{\sin(y)}{1+3\cos^2(y)}dy - \Omega \\ &\Rightarrow \Omega = \frac{\pi+4}{2} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{1+u^2} \frac{du}{\sqrt{3}}, u = \sqrt{3}\cos(y) \\ &= \frac{\pi+4}{2\sqrt{3}} \arctan(u) \Big|_{-\sqrt{3}}^{\sqrt{3}} = \frac{\pi+4}{2\sqrt{3}} \frac{2\pi}{3} = \frac{\pi(\pi+4)}{3\sqrt{3}}\end{aligned}$$

Therefore the solution is:

$$\Omega = \frac{\pi(\pi+4)}{3\sqrt{3}}.$$

□

Solutoin 8 by Jose Luis Diaz-Barrero, Barcelona, Spain.

To evaluate

$$\Omega = \int_0^{\pi} \frac{(x+2)\sin x}{1+3\cos^2 x}dx.$$

we make the substitution $x \rightarrow \pi - x$. Then

$$\Omega = \int_0^{\pi} \frac{(\pi-x+2)\sin x}{1+3\cos^2 x}dx.$$

Add the two expressions for Ω yields

$$2\Omega = \int_0^{\pi} \frac{(x+2) + (\pi-x+2)}{1+3\cos^2 x} \sin x dx = \int_0^{\pi} \frac{x+4}{1+3\cos^2 x} \sin x dx$$

Thus $\Omega = \frac{\pi+4}{2} \int_0^\pi \frac{\sin x}{1+3\cos^2 x} dx$. Now, we evaluate

$$I = \int_0^\pi \frac{\sin x}{1+3\cos^2 x} dx$$

Using the substitution $u = \cos x$, $du = -\sin x dx$, we have

$$I = \int_1^{-1} \frac{-1}{1+3u^2} du = \int_{-1}^1 \frac{1}{1+3u^2} du.$$

Since the integrand is even, then

$$I = 2 \int_0^1 \frac{1}{1+3u^2} du.$$

Since $\int \frac{1}{1+3u^2} du = \frac{1}{\sqrt{3}} \arctan(\sqrt{3}u)$, then

$$I = \frac{2}{\sqrt{3}} \arctan(\sqrt{3}) = \frac{2\pi}{3\sqrt{3}}$$

on account that $\arctan(\sqrt{3}) = \frac{\pi}{3}$. Finally,

$$\Omega = \frac{\pi+4}{2} I = \frac{\pi+4}{2} \cdot \frac{2\pi}{3\sqrt{3}} = \frac{\pi(\pi+4)}{3\sqrt{3}}$$

□

Solution 9 by Luke Paluso, Blinn College, Brenham, TX.

Using the substitution method (U) with $u = \sqrt{3}\cos x$ and properties of definite integrals one has

$$\begin{aligned} \Omega &= \int_0^\pi \frac{(x+2)\sin x}{1+(\sqrt{3}\cos x)^2} dx \\ &=^U \frac{1}{\sqrt{3}} \int_{-\sqrt{3}}^{\sqrt{3}} \frac{\arccos(\frac{u}{\sqrt{3}}) + 2}{1+u^2} du \\ &= \frac{1}{\sqrt{3}} \left[\int_{-\sqrt{3}}^{\sqrt{3}} \frac{\arccos(\frac{u}{\sqrt{3}})}{1+u^2} du + \int_{-\sqrt{3}}^{\sqrt{3}} \frac{2}{1+u^2} du \right]. \end{aligned}$$

Do the substitution $u = \tan \theta(A)$, using the identity $\arccos(y) + \arcsin(y) = \pi/2(B)$, the Maclaurin series for arcsine (C), and the fact that $\tan^{2n+1} \theta$ is odd for $n \in \mathbb{N}(D)$, one evaluates the 1st integral:

$$\begin{aligned} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\arccos(\frac{u}{\sqrt{3}})}{1+u^2} du &=^A \int_{-\sqrt{3}}^{\sqrt{3}} \arccos\left(\frac{1}{\sqrt{3}} \tan \theta\right) d\theta \\ &=^B \frac{\pi}{2} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\theta - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \arcsin\left(\frac{1}{\sqrt{3}} \tan \theta\right) d\theta \\ &=^C \frac{\pi}{2} \cdot \frac{2\pi}{3} - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2 (2n+1)} \left(\frac{1}{\sqrt{3}} \tan \theta\right)^{2n+1} d\theta \\ &= \frac{\pi^2}{3} - \sum_{n=0}^{\infty} \frac{(2n)!}{(2^n n!)^2 (2n+1) \sqrt{3}^{2n+1}} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \tan^{2n+1} \theta d\theta \\ &=^D \frac{\pi^2}{3} \end{aligned}$$

The 2nd integral is straight forward

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{2}{1+u^2} du = 2 \arctan u \Big|_{-\sqrt{3}}^{\sqrt{3}} = 2 \cdot (\arctan \sqrt{3} - \arctan(-\sqrt{3})) = \frac{4\pi}{3}.$$

Substituting these into the original chain of equalities yields the answer:

$$\Omega = \frac{1}{\sqrt{3}} \left(\frac{\pi^2}{3} + \frac{4\pi}{3} \right) = \frac{\pi^2 + 4\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi^2 + 4\sqrt{3}\pi}{9}.$$

□

Solution 10 by Michel Bataille, Rouen, France.

We calculate

$$\Omega_1 = \int_0^\pi \frac{\sin x}{1+3\cos^2 x} dx \text{ and } \Omega_2 = \int_0^\pi \frac{x \sin x}{1+3\cos^2 x} dx.$$

First, the substitution $u = \cos x$ leads to

$$\Omega_1 = \int_{-1}^1 \frac{du}{1+3u^2} = \frac{1}{\sqrt{3}} \left[\arctan(u\sqrt{3}) \right]_{-1}^1 = \frac{2\pi}{3\sqrt{3}}$$

Then we have

$$\Omega_2 = \int_0^\pi \frac{(\pi - u) \sin u}{1+3\cos^2 u} du = \pi \cdot \Omega_1 - \Omega_2$$

hence $\Omega_2 = \frac{\pi}{2} \cdot \Omega_1 = \frac{\pi^2}{3\sqrt{3}}$. Finally, we obtain

$$\Omega = 2\Omega_1 + \Omega_2 = \frac{\pi(\pi + 4)}{3\sqrt{3}}.$$

□

Solution 11 by Perfetti Paolo, dipartimento di matematica Universita di "Tor Vergata", Roma, Italy

The integral is

$$\begin{aligned} \Omega &= \int_0^\pi \frac{x \sin x}{1+3\cos^2 x} dx + \int_0^x \frac{2 \sin x}{1+3\cos^2 x} dx = \Omega_1 + \Omega_2 \\ \Omega_2 &= \int_0^\pi \frac{\sin x}{1-i\sqrt{3}\cos x} dx + \int_0^\pi \frac{\sin x}{1+i\sqrt{3}\cos x} dx = \frac{1}{i\sqrt{3}} \operatorname{Ln} \frac{1-i\sqrt{3}\cos x}{1+i\sqrt{3}\cos x} \Big|_0^\pi = \\ &= \frac{1}{i\sqrt{3}} \operatorname{Ln} \frac{1+i\sqrt{3}}{1-i\sqrt{3}} - \frac{1}{i\sqrt{3}} \operatorname{Ln} \frac{1-i\sqrt{3}}{1+i\sqrt{3}} = \frac{2}{i\sqrt{3}} \operatorname{Ln} \frac{1+i\sqrt{3}}{1-i\sqrt{3}} \\ &= \frac{2}{i\sqrt{3}} \operatorname{Ln} \frac{e^{i\pi/3}}{e^{-i\pi/3}} = \frac{2}{i\sqrt{3}} \frac{2i\pi}{3} = \frac{4\pi}{3\sqrt{3}} \\ \Omega_1 &= \frac{1}{2} \int_{-\pi}^\pi \frac{x \sin x}{1+3\cos^2 x} dx = \frac{1}{2} \int_{-\pi}^0 \frac{x \sin x}{1+3\cos^2 x} dx + \frac{1}{2} \int_0^\pi \frac{x \sin x}{1+3\cos^2 x} dx = \\ &= \frac{1}{2} \int_0^\pi \frac{-(x-\pi) \sin x}{1+3\cos^2 x} dx + \frac{1}{2} \int_0^\pi \frac{x \sin x}{1+3\cos^2 x} dx = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1+3\cos^2 x} dx = \\ &= \frac{2\pi}{3\sqrt{3}} \frac{\pi}{2} = \frac{\pi^2}{3\sqrt{3}} \end{aligned}$$

Then result is $\frac{\pi(4+\pi)}{3\sqrt{3}}$. Now consider this: $\frac{d}{da} \int_0^\infty \frac{x^a dx}{x^4+1} \Big|_{a=3/2}$. By passing to the complex variable z , it follows that

$$\begin{aligned} \int_0^\infty \frac{x^a dx}{x^4+1} &= \frac{2\pi i}{1-e^{2i\pi a}} \sum_{k=0}^3 \operatorname{Res} \frac{z^a}{z^4+1} \Big|_{a=e^{\frac{i(1+3k)\pi}{4}}} = \frac{2\pi i}{1-e^{2i\pi a}} \sum_{k=0}^3 \frac{1}{4} e^{\frac{i\pi(1-2k)(a-3)}{4}} = \\ &= \frac{\pi}{2(1-e^{2i\pi a})} e^{\frac{i\pi(a-3)}{4}} \sum_{k=0}^3 e^{\frac{i\pi k(a-3)}{2}} = \frac{\pi}{2(1-e^{2i\pi a})} e^{\frac{i\pi(a-3)}{4}} \frac{1-e^{i\pi 2k(a-3)}}{1-e^{\frac{i\pi(a-3)}{2}}} = \\ &= \frac{\pi i}{2} \frac{e^{\frac{i\pi(a-3)}{4}}}{1-e^{\frac{i\pi(a-3)}{2}}} = \frac{\pi}{2} \frac{1}{-2i \sin \frac{\pi(a-3)}{4}} \end{aligned}$$

Doing the derivative and evaluating at $a = 3/2$ we get

$$\frac{\pi}{4} \frac{\cos \frac{\pi(a-3)}{4}}{\sin^2 \frac{\pi(a-3)}{4}} \Big|_{a=\frac{3}{2}} = \frac{\pi^2 \cos \frac{3\pi}{8}}{16 \sin^2 \frac{3\pi}{8}} = \frac{\pi^2 \sin \frac{\pi}{8}}{16 \cos^2 \frac{\pi}{8}} = \frac{\pi^2 \frac{\sqrt{2-\sqrt{2}}}{2}}{16 \frac{2+\sqrt{2}}{4}} = \frac{\pi^2}{2} \frac{(2-\sqrt{2})^{3/2}}{8} = \frac{\pi^2}{2} \sin^3 \frac{\pi}{8}.$$

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