

5825. Let $f(n)_{n=0}^{\infty}$ be the Fibonacci sequence; i.e., $f_0 = 0, f_1 = 1, f_{n+2} = f_{n+1} + f_n$ for $n \geq 0$. Prove that for all $n \geq 1$:

$$8 \left(\sum_{k=1}^n f_k \sin \frac{1}{k} \right) \left(\sum_{k=1}^n f_k \cos \frac{1}{k} \right) < n f_{n+2}^2$$

Proposed by Daniel Sitaru - Romania

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By the AM-GM inequality,

$$2 \left(\sum_{k=1}^n f_k \sin \frac{1}{k} \right) \left(\sum_{k=1}^n f_k \cos \frac{1}{k} \right) \leq \left(\sum_{k=1}^n f_k \sin \frac{1}{k} \right)^2 + \left(\sum_{k=1}^n f_k \cos \frac{1}{k} \right)^2.$$

Now, by the Cauchy - Schwartz inequality,

$$\begin{aligned} \left(\sum_{k=1}^n f_k \sin \frac{1}{k} \right)^2 &\leq \left(\sum_{k=1}^n f_k^2 \right) \cdot \left(\sum_{k=1}^n \sin^2 \frac{1}{k} \right), \\ \left(\sum_{k=1}^n f_k \cos \frac{1}{k} \right)^2 &\leq \left(\sum_{k=1}^n f_k^2 \right) \cdot \left(\sum_{k=1}^n \cos^2 \frac{1}{k} \right). \end{aligned}$$

From where, the left-hand side of the proposed inequality, say *LHS* verifies

$$LHS \leq 4 \left(\sum_{k=1}^n f_k^2 \right) \cdot \left(\sum_{k=1}^n \sin^2 \frac{1}{k} + \sum_{k=1}^n \cos^2 \frac{1}{k} \right) = 4n f_n f_{n+1}$$

because $\sin^2 \frac{1}{k} + \cos^2 \frac{1}{k} = 1$ and $\sum_{k=1}^n f_k^2 = f_n f_{n+1}$.

Since $4f_n f_{n+1} < (f_n + f_{n+1})^2 = f_{n+2}^2$ by the AM-GM inequality, the conclusion follows. $\square$

Solution 2 by Brian D. Beasley, Simpsonville, SC.

For each $n \geq 1$, let $S_n = \sum_{k=1}^n f_k \sin \frac{1}{k}$ and $C_n = \sum_{k=1}^n f_k \cos \frac{1}{k}$.

Since $\sum_{k=1}^n f_k = f_{n+2} - 1$, we have

$$S_n < f_{n+2} - 1 \text{ and } C_n < f_{n+2} - 1.$$

For $n \in \{1, 2\}$, direct calculations show that

$$8S_n C_n < n f_{n+2}^2.$$

For $n \in \{3, 4, 5, 6, 7\}$, direct calculations show that

$$8S_n C_n < 8S_n (f_{n+2} - 1) < n f_{n+2}^2.$$

For $n \geq 8$, we have

$$8S_n C_n < 8(f_{n+2} - 1)^2 < 8f_{n+2}^2 \leq n f_{n+2}^2.$$

Addendum. Calculations:

n	$8S_n C_n$	$8S_n(f_{n+2} - 1)$	nf_{n+2}^2
1	3.637	(6.732)	4
2	14.983	(21.134)	18
3	52.271	63.209	75
4	135.104	152.180	256
5	329.964	356.241	845
6	765.940	806.082	2646
7	1756.468	1818.655	8092

□

Solution 3 by Jose Luis Diaz-Barrero, Barcelona, Spain.

Let $A = \sum_{k=1}^n f_k \sin \frac{1}{k}$ and $B = \sum_{k=1}^n f_k \cos \frac{1}{k}$.

From $(A - B)^2 \geq 0$ we obtain $8AB \leq 4(A^2 + B^2)$ and

$$\begin{aligned}
 A^2 + B^2 &= \sum_{k=1}^n f_k^2 \sin^2 \frac{1}{k} + \sum_{k=1}^n f_k^2 \cos^2 \frac{1}{k} \\
 &+ 2 \sum_{1 \leq i < j \leq n} f_i f_j \left(\sin \frac{1}{i} \sin \frac{1}{j} + \cos \frac{1}{i} \cos \frac{1}{j} \right) \\
 &= \sum_{k=1}^n f_k^2 \left(\sin^2 \frac{1}{k} + \cos^2 \frac{1}{k} \right) + 2 \sum_{1 \leq i < j \leq n} f_i f_j \left(\sin \frac{1}{i} \sin \frac{1}{j} + \cos \frac{1}{i} \cos \frac{1}{j} \right)
 \end{aligned}$$

Using the well-known formulae $\sin^2 x + \cos^2 x = 1$ and $\sin a \sin b + \cos a \cos b = \cos(a - b)$, we get

$$A^2 + B^2 = \sum_{k=1}^n f_k^2 + 2 \sum_{1 \leq i < j \leq n} f_i f_j \cos \left(\frac{1}{i} - \frac{1}{j} \right) \leq \sum_{k=1}^n f_k^2 + 2 \sum_{1 \leq i < j \leq n} f_i f_j.$$

Thus

$$A^2 + B^2 \leq \left(\sum_{k=1}^n f_k \right)^2, \text{ and therefore } 8AB \leq 4 \left(\sum_{k=1}^n f_k \right)^2.$$

Since $\sum_{k=1}^n f_k = f_{n+2} - 1$, as it is well-known, then $8AB \leq 4(f_{n+2} - 1)^2$. So it suffices to show

$$4(f_{n+2} - 1)^2 < nf_{n+2}^2 \Leftrightarrow 4 \left(1 - \frac{1}{f_{n+2}} \right)^2 < n.$$

To do that we will distinguish the following cases:

- For $n = 1$ we have $8 \sin 1 \cos 1 = 4 \sin 2 < 4 = f_3^2$.
- For $n = 2$ we have $8 \left(\sin 1 + \sin \frac{1}{2} \right) \left(\cos 1 + \cos \frac{1}{2} \right) < 18 = 2f_4^2$
- For $n = 3$ we have $8 \left(\sin 1 + \sin \frac{1}{2} + 2 \sin \frac{1}{3} \right) \left(\cos 1 + \cos \frac{1}{2} + 2 \cos \frac{1}{3} \right) < 75 = 3f_5^2$
- For $n \geq 4$, we have $f_{n+2} \geq 3$, so

$$0 < 1 - \frac{1}{f_{n+2}} < 1 \Rightarrow 4 \left(1 - \frac{1}{f_{n+2}} \right)^2 < 4 \leq n$$

Thus the inequality holds for all $n \geq 4$.

Finally, we conclude that the inequality holds for every $n \geq 1$.

□

Solution 4 by Michael Bataille, Rouen, France.

Let $S_n = \sum_{k=1}^n f_k \sin \frac{1}{k}$ and $C_n = \sum_{k=1}^n f_k \cos \frac{1}{k}$. The general inequality $4xy \leq (x+y)^2$ yields

$$(1) \quad 4S_n C_n \leq \left(\sum_{k=1}^n f_k \left(\sin \frac{1}{k} + \cos \frac{1}{k} \right) \right)^2$$

and the Cauchy - Schwarz inequality gives

$$(2) \quad \left(\sum_{k=1}^n f_k \left(\sin \frac{1}{k} + \cos \frac{1}{k} \right) \right)^2 \leq \left(\sum_{k=1}^n f_k^2 \right) \left(\sum_{k=1}^n \left(\sin \frac{1}{k} + \cos \frac{1}{k} \right)^2 \right)$$

Now, we have $\sum_{k=1}^n f_k^2 = f_n f_{n+1}$ (by an easy induction) and

$$\sum_{k=1}^n \left(\sin \frac{1}{k} + \cos \frac{1}{k} \right)^2 = \sum_{k=1}^n \left(1 + \sin \frac{2}{k} \right) < 2n$$

(since $\sin \frac{2}{k} < 1$ for $k = 1, 2, \dots, n$). From (1) and (2), we now deduce

$$8S_n C_n < \frac{1}{2} \cdot 4f_n f_{n+1} \cdot 2n \leq n(f_n + f_{n+1})^2 = n f_{n+2}^2$$

and the desired inequality $8S_n C_n < n f_{n+2}^2$ follows. $\square$

Solution 5 by Ulrich Abel, Technische Hochschule Mittelhessen, Friedberg, Germany.

Taking advantage of the well-known formula $\sum_{k=1}^n f_k = f_{n+2} - 1$ and the obvious relations $0 \leq \sin \frac{1}{k} \leq \sin 1 \approx 0.84$ and $0 \leq \cos \frac{1}{k} \leq 1$ for all $k \geq 1$, we conclude that

$$\left(\sum_{k=1}^n f_k \sin \frac{1}{k} \right) \left(\sum_{k=1}^n f_k \cos \frac{1}{k} \right) \leq (\sin 1) \left(\sum_{k=1}^n f_k \right) \left(\sum_{k=1}^n f_k \right) = (\sin 1)(f_{n+2} - 1)^2 \leq (\sin 1) f_{n+2}^2$$

Observing that $8 \sin 1 \approx 6.73$ we see that this implies the inequality of Problem #5825, for all $n \geq 7$. The finitely many cases $n \leq 6$ follow by direct computation. Remark. The quantity

$$S_n := \left(\sum_{k=1}^n f_k \sin \frac{1}{k} \right) \left(\sum_{k=1}^n f_k \cos \frac{1}{k} \right)$$

satisfies the (rough) asymptotic relation

$$\log S_n^{\frac{1}{n}} \sim 2 \log \phi \quad (n \rightarrow \infty),$$

where $\phi = (1 + \sqrt{5})/2$ is the Golden Ratio. From

$$\begin{aligned} \left(\sin \frac{1}{n} \right) (\cos 1) \left(1 - \frac{1}{f_{n+2}} \right)^2 f_{n+2}^2 &\leq \left(\sin \frac{1}{n} \right) (\cos 1) \left(\sum_{k=1}^n f_k \right)^2 \leq S_n \\ &\leq (\sin 1) \left(\cos \frac{1}{n} \right) \left(\sum_{k=1}^n f_k \right)^2 \leq f_{n+2}^2 \end{aligned}$$

it follows that

$$a(n) + \frac{1}{n} \log^2 f_{n+2} \leq \frac{1}{n} \log S_n \leq \frac{1}{n} \log f_{n+2}^2,$$

where

$$a(n) = \frac{1}{n} \log \left(\sin \frac{1}{n} \right) + \frac{1}{n} \log(\cos 1) + \frac{1}{n} \log \left(1 - \frac{1}{f_{n+2}} \right)^2 = o(1)$$

as $n \rightarrow \infty$. For the first term we used that $\left| \log \left(\sin \frac{1}{n} \right) \right| \sim \log n = o(n)$ as $n \rightarrow \infty$.

Therefore,

$$\frac{1}{n} \log S_n = \frac{1}{n} \log f_{n+2}^2 + o(1) \quad (n \rightarrow \infty).$$

It is a consequence of the Moivre-Binet representation that $f_n \sim \phi^n / \sqrt{5}$ as $n \rightarrow \infty$. We infer that

$$\frac{1}{n} \log S_n = \frac{1}{n} \log \phi^{2n+2} - \frac{1}{n} \log \sqrt{5} + O(1) = \frac{2n+2}{n} \log \phi + o(1) = 2 \log \phi + o(1)$$

as $n \rightarrow \infty$, which proves our claim. $\square$

Solution 6 by proposer.

By induction we prove that:

$$(1) \quad f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1}$$

$$f_2 = f_0 + f_1 = 0 + 1 = 1$$

$$\text{For } n = 1 \text{ in (1) : } f_1^2 = f_1 f_2 \Leftrightarrow 1^2 = 1 \cdot 1 \text{ (true)}$$

$$P(n) : f_1^2 + f_2^2 + \dots + f_n^2 = f_n f_{n+1} \text{ (suppose true)}$$

$$P(n+1) : f_1^2 + f_2^2 + \dots + f_n^2 + f_{n+1}^2 = f_{n+1} f_{n+2} \text{ (to prove)}$$

$$\begin{aligned} f_1^2 + f_2^2 + \dots + f_n^2 + f_{n+1}^2 &\stackrel{P(n)}{=} f_n f_{n+1} + f_{n+1}^2 = \\ &= f_{n+1} (f_n + f_{n+1}) = f_{n+1} \cdot f_{n+2} \end{aligned}$$

$$P(n) \rightarrow P(n+1)$$

$$\left(\sum_{k=1}^n f_k \sin \frac{1}{k} \right) \cdot \left(\sum_{k=1}^n f_k \cos \frac{1}{k} \right) \leq \left| \sum_{k=1}^n f_k \sin \frac{1}{k} \right| \cdot \left| \sum_{k=1}^n f_k \cos \frac{1}{k} \right| \stackrel{\text{CBS}}{\leq}$$

$$\leq \sqrt{\sum_{k=1}^n f_k^2} \cdot \sqrt{\sum_{k=1}^n \sin^2 \frac{1}{k}} \cdot \sqrt{\sum_{k=1}^n f_k^2} \cdot \sqrt{\sum_{k=1}^n \cos^2 \frac{1}{k}} =$$

$$= \sum_{k=1}^n f_k^2 \cdot \sqrt{\left(\sum_{k=1}^n \sin^2 \frac{1}{k} \right) \left(\sum_{k=1}^n \cos^2 \frac{1}{k} \right)} \leq$$

$$\stackrel{\text{AM-GM}}{\leq} \frac{\sum_{k=1}^n \sin^2 \frac{1}{k} + \sum_{k=1}^n \cos^2 \frac{1}{k}}{2} \cdot \sum_{k=1}^n f_k^2 =$$

$$\stackrel{(1)}{=} \frac{\sum_{k=1}^n (\sin^2 \frac{1}{k} + \cos^2 \frac{1}{k})}{2} \cdot f_n f_{n+1} = \frac{n}{2} f_n f_{n+1} <$$

$$\stackrel{\text{AM-GM}}{<} \frac{n}{2} \cdot \left(\frac{f_n + f_{n+1}}{2} \right)^2 = \frac{n}{8} f_{n+2}^2$$

$$\frac{8}{n} \left(\sum_{k=1}^n f_k \sin \frac{1}{k} \right) \left(\sum_{k=1}^n f_k \cos \frac{1}{k} \right) < f_{n+2}^2$$

$\square$

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