5823

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5823. Prove that if $0 < a \le b$, then:

$$\left(\int_a^b e^{-x^2}dx\right)^2 \ge \left(\int_{\frac{a+b}{2}}^b e^{-x^2}dx + \int_{\sqrt{ab}}^b e^{-x^2}dx\right)\left(\int_a^{\frac{a+b}{2}} e^{-x^2}dx + \int_a^{\sqrt{ab}} e^{-x^2}dx\right)$$

Daniel Sitaru

Solution 1 by Sri Hari Bhupala Haribhakta, Bengaluru, India.
Let $f(x) = e^{-x^2}$ and set

$$m = \frac{a+b}{2}, g = \sqrt{ab}.$$

Although the integral is Gaussian, the inequality holds for any integrable function f; it is a consequence of splitting $\int_a^b f(x)$ at the two points m and g.
Since $0 < a \le b$, we have $a \le g \le b$, hence all integrals below are well-defined.
Define

$$A = \int_a^m f(x)dx, \quad B = \int_m^b f(x)dx, \quad C = \int_a^g f(x)dx, \quad D = \int_g^b f(x)dx.$$

Then

$$\int_a^b f(x)dx = A + B = C + D$$

Let

$$I := \int_a^b f(x)dx, \qquad S := A + C.$$

Using $B = I - A$ and $D = I - C$, we obtain

$$B + D = (I - A) + (I - C) = 2I - (A + C) = 2I - S.$$

Therefore the right-hand side of the desired inequality equals

$$\left(\int_m^b f(x)dx+\int_g^b f(x)dx\right)\left(\int_a^m f(x)dx+\int_a^g f(x)dx\right)= (B+D)(A+C) = (2I-S)S.$$

Hence the inequality becomes

$$I^2 \ge (2I - S)S \Leftrightarrow I^2 - 2IS + S^2 \ge 0 \Leftrightarrow (I - S)^2 \ge 0,$$

which is always true. This proves the required inequality. □

Solution 2 by Michel Bataille, Rouen, France.
Let

$$X = \int_{\frac{a+b}{2}}^b e^{-x^2}dx + \int_{\sqrt{ab}}^b e^{-x^2}dx, \qquad Y = \int_a^{\frac{a+b}{2}} e^{-x^2}dx + \int_a^{\sqrt{ab}} e^{-x^2}dx.$$

Then, we have

$$X+Y=\int_a^{\frac{a+b}{2}} e^{-x^2}dx+\int_{\frac{a+b}{2}}^b e^{-x^2}dx+\int_a^{\sqrt{ab}} e^{-x^2}dx+\int_{\sqrt{ab}}^b e^{-x^2}dx$$

$$=\int_a^b e^{-x^2}dx+\int_a^b e^{-x^2}dx=2\int_a^b e^{-x^2}dx$$

From the obvious inequality $(X+Y)^2\geq 4XY$, we deduce $4\Big(\int_a^b e^{-x^2}dx\Big)^2\geq 4XY$, hence

$$\left(\int_a^b e^{-x^2}dx\right)^2\geq XY,$$

as desired. □

Solution 3 by Prakash Pant, The University of Vermont, Bardiya, Nepal.
Let $F(x)$ be the real antiderivative of e^{-x^2}. Then the given problem can be restated as:

$$[F(b)-F(a)]^2\geq\Big[F(b)-F\Big(\frac{a+b}{2}\Big)+F(b)-F(\sqrt{ab})\Big]\Big[F\Big(\frac{a+b}{2}\Big)-F(a)+F(\sqrt{ab})-F(a)\Big]$$

$$[F(b)-F(a)]^2\geq\Big[2F(b)-F\Big(\frac{a+b}{2}\Big)-F(\sqrt{ab})\Big]\Big[F\Big(\frac{a+b}{2}\Big)+F(\sqrt{ab})-2F(a)\Big]$$

$$F^2(b)-2F(b)F(a)+F^2(a)\geq 2F(b)\Big[F\Big(\frac{a+b}{2}\Big)+F(\sqrt{ab})\Big]-4F(b)F(a)-$$

$$-\Big[F\Big(\frac{a+b}{2}\Big)+F(\sqrt{ab})\Big]^2+2F(a)\Big[F\Big(\frac{a+b}{2}\Big)+F(\sqrt{ab})\Big]$$

which can be rewritten as

$$F(b)+F(a)]^2+\Big[F\Big(\frac{a+b}{2}\Big)+F(\sqrt{ab})\Big]^2\geq 2\Big[F\Big(\frac{a+b}{2}\Big)+F(\sqrt{ab})\Big][F(b)+F(a)]$$

$$\Big[F(b)+F(a)-F\Big(\frac{a+b}{2}\Big)-F(\sqrt{ab})\Big]^2\geq 0$$

Since square of a real number cannot be negative, the given statement is true which proves the original statement we began with, hence concluding the proof. □

Solution 4 by Albert Stadler, Herrliberg, Switzerland.
We prove more generally that

$$\left(\int_a^b f(x)dx\right)^2\geq\left(\int_u^b f(x)dx+\int_v^b f(x)dx\right)\left(\int_a^u f(x)dx+\int_a^v f(x)dx\right)$$

for any integrable function f and any $u,v\in[a,b]$. Let

$$A:\int_a^b f(x)dx, B:=\int_u^b f(x)dx, C:=\int_v^b f(x)dx$$

Then above inequality reads as

$$A^2\geq(B+C)(A-B+A-C)$$

which is equivalent to $(A-B-C)^2\geq 0$ which is obviously true. □

Solution 5 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
Subintegral function, e^{-x^2} is positive. The solution follows by several applications of the GM-AM inequality:
Since

$$\sqrt{ab} \leq \frac{a+b}{2},$$

then

$$\int_{\frac{a+b}{2}}^{b} e^{-x^2}dx + \int_{\sqrt{ab}}^{b} e^{-x^2}dx \leq 2\int_{\frac{a+b}{2}}^{b} e^{-x^2}dx$$

and

$$\int_{a}^{\frac{a+b}{2}} e^{-x^2}dx + \int_{a}^{\sqrt{ab}} e^{-x^2}dx \leq 2\int_{a}^{\frac{a+b}{2}} e^{-x^2}dx.$$

Therefore, for the right-hand side (RHS) of the proposed inequality we have

$$RHS \leq 4\left(\int_{\frac{a+b}{2}}^{b} e^{-x^2}dx\right)\left(\int_{a}^{\frac{a+b}{2}} e^{-x^2}dx\right)$$

Again, by the GM-AM inequality,

$$RHS \leq 4\frac{\left(\int_{a}^{\frac{a+b}{2}} e^{-x^2}dx + \int_{\frac{a+b}{2}}^{b} e^{-x^2}dx\right)^2}{4} = \left(\int_{a}^{b} e^{-x^2}dx\right)^2.$$

□

Solution 6 by proposer.

$$0 < a \leq \sqrt{ab} \leq \frac{a+b}{2} \leq b$$

Denote:

$$u = \int_{a}^{\sqrt{ab}} e^{-x^2}dx; v = \int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}dx; w = \int_{\frac{a+b}{2}}^{b} e^{-x^2}dx$$

$$\int_{\sqrt{ab}}^{b} e^{-x^2}dx = \int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}dx + \int_{\frac{a+b}{2}}^{b} e^{-x^2}dx = v + w$$

$$\int_{a}^{\frac{a+b}{2}} e^{-x^2}dx = \int_{a}^{\sqrt{ab}} e^{-x^2}dx + \int_{\sqrt{ab}}^{\frac{a+b}{2}} e^{-x^2}dx = u + v$$

Inequality to prove can be written:

$$(u+v+w)^2 \geq (w+v+w)(u+v+u)$$

$$(u+v+w)^2 \geq (v+2w)(v+2u)$$

$$u^2+v^2+w^2+2uv+2uw+2vw \geq v^2+2uv+2vw+4wu$$

$$u^2+w^2-2uw \geq 0 \Leftrightarrow (u-w)^2 \geq 0$$

Equality holds for $a = b$. □

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