**5816**

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5816. For $0 < a \leq b$, prove:

$$\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}dx \leq \frac{1}{2}\tan^{-1}\Big(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\Big).$$

*Daniel Sitaru*

*Solution 1 by Saurab Banstola, Gandaki Boarding School, Pokhara, Nepal.*
We begin be rewriting the right-hand side. Observe that

$$\frac{1}{2}\tan^{-1}\Big(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\Big) = \frac{1}{2}(\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})).$$

This difference can be expressed as an integral:

$$\frac{1}{2}\tan^{-1}(e^{x^2})\Big|_a^b = \frac{1}{2}\int_a^b \frac{d}{dx}(\tan^{-1}(e^{x^2})x = \int_a^b \frac{xe^{x^2}}{1+e^{2x^2}}dx$$

Thus, the original inequality reduces to comparing the two integrands:

$$\frac{xe^{e^{x^2}}\sqrt{e^{x^2}}}{e^{3x^2}+1} \leq \frac{xe^{x^2}}{1+e^{2x^2}} \Leftrightarrow \frac{e^{\frac{x^2}{2}}}{e^{3x^2}+1} \leq \frac{1}{1+e^{2x^2}}.$$

Since all quantities involved are positive, we may cross-multiply safely, obtaining the equivalent inequality

$$e^{\frac{x^2}{2}} + e^{\frac{5x^2}{2}} \leq 1 + e^{3x^2} \tag{1}$$

To justify (1), we apply Karamata's inequality. Consider the two sequences

$$(3,0) \qquad \text{and} \qquad \Big(\frac{5}{2},\frac{1}{2}\Big).$$

It is easy to check that

$$(3,0) \succ \Big(\frac{5}{2},\frac{1}{2}\Big),$$

meaning that the former majorizes the latter. Because the function

$$f(t) = e^{tx^2}$$

is convex for every fixed $x$ (as an exponential function), Karamata's inequality yields

$$f(3) + f(0) \geq f\Big(\frac{5}{2}\Big) + f\Big(\frac{1}{2}\Big).$$

Substituting back $f(t) = e^{tx^2}$ gives exactly inequality (1). This verifies the requiered comparison of integrands, and therefore the given integral inequality follows. $\square$

*Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.*
Let $I$ denote the integral. Then

$$I = \frac{1}{3}\int_a^b \frac{3xe^{\frac{3}{2}x^2}}{(e^{\frac{3}{2}x^2})^2+1}dx = \frac{1}{3}\tan^{-1} e^{\frac{3}{2}x^2}\Big]_a^b = \frac{1}{3}\tan^{-1} e^{\frac{3}{2}b^2} - \frac{1}{3}\tan^{-1} e^{\frac{3}{2}a^2}.$$

Also, $\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right) = \tan^{-1} e^{b^2} - \tan^{-1} e^{a^2}$. Let us change variables by doing $x = e^{a^2}, y = e^{b^2}$.
So the inequality to be proved reads now as:
For $1 < x \leq y$, prove

$$\frac{1}{2}\tan^{-1} x - \frac{1}{3}\tan^{-1} x^{\frac{3}{2}} \leq \frac{1}{2}\tan^{-1} y - \frac{1}{3}\tan^{-1} y^{\frac{3}{2}}.$$

Let $f$ be the function defined by $f(x) = \frac{1}{2}\tan^{-1} x - \frac{1}{3}\tan^{-1} x^{\frac{3}{2}}$, for $x > 1$. Then, for $x > 1$:

$$f'(x) = \frac{(-1+\sqrt{x})^2(1+\sqrt{x}+x+x^{\frac{3}{2}+x^2})}{2(1+x)(1+x^2)(1-x+x^2)} > 0,$$

so function $f$ is increasing for $x > 1$ and the problem is done. □

*Solution 3 by Ṕeter Fülöp, Gyömrő, Hungary.*
1. Regarding the left hand side:
(i) - Substitution $x^2 = t$:
(ii) - Substitution $e^t = z$:
(iii) - Substitution $z^3 = -r$:

$$\frac{1}{2}\int_{a^2}^{b^2} \frac{e^{\frac{3t}{2}}}{1+e^{3t}}dt = \frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \frac{z^{\frac{1}{2}}}{1+z^3}dz = \frac{i}{6}\int_{-e^{3b^2}}^{-e^{3a^2}} \frac{r^{-\frac{1}{2}}}{1-r}dr$$

(iv) - Introduction of the incomplete $\beta$ function:

$$\frac{i}{6}\beta - e^{3a^2}\left(\frac{1}{2}, 0\right) - \frac{i}{6}\beta - e^{3b^2}\left(\frac{1}{2}, 0\right)$$

(v) - Applied the summation form of incomplete $\beta$ function:

$$\frac{i}{6}\sum_{k=0}^{\infty}\frac{(-e^{3a^2})^{(k+\frac{1}{z})}}{k+\frac{1}{2}} - \frac{i}{6}\sum_{k=0}^{\infty}\frac{(-e^{3b^2})^(k+\frac{1}{z})}{k+\frac{1}{2}} =$$

$$-\frac{1}{3}\sum_{k=0}^{\infty}\frac{(-1)^k(e^{\frac{3a^2}{2}})^{(2k+1)}}{2k+1} + \frac{1}{3}\sum_{k=0}^{\infty}\frac{(-1)^k(e^{\frac{3b^2}{2}})^(2k+1)}{2k+1}$$

(vi) - Known that $\tan^{-1}(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}$ we get:

$$\frac{1}{3}\tan^{-1}(e^{\frac{3b^2}{2}}) - \frac{1}{3}\tan^{-1}(e^{\frac{3a^2}{2}})$$

(vii) - Transformation back to integral form,
(viii) - Substitution $x = \frac{3t}{2}$.

$$LHS = \frac{1}{3}\int_{e^{\frac{3a^2}{2}}}^{e^{\frac{3b^2}{2}}} \frac{1}{1+x^2}dx = \underline{\frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \frac{1}{1+(\frac{3t}{2})^2}dt}$$

2. Regarding the left hand side:
(i) - Application of the following identity: $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}(x) - \tan^{-1}(y)$
In this case

$$RHS = \frac{1}{2}(\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})) = \underline{\frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \frac{1}{1+t^2}dt}$$

3. The following inequality is true for the integrands:

$$0 \leq \frac{1}{1+(\frac{3t}{2})^2} \leq \frac{1}{1+t^2} \text{ for } \forall t$$

So it is also true for it's integrals in the same domain:

$$\frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \frac{1}{1+(\frac{3t}{2})^2}dt \leq \frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \frac{1}{1+t^2}dt$$

Statement is proved. □

*Solution 4 by Prakash Pant, The University of Vermont, Bardiya, Nepal.*
Observe that:

$$\frac{1}{2}\tan^{-1}\Big(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\Big) = \frac{1}{2}(\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})) = \frac{1}{2}(\tan^{-1}(e^{x^2}))|_a^b$$

$$= \frac{1}{2}\int_a^b \frac{d}{dx}(\tan^{-1}(e^{x^2}))dx = \int_a^b \frac{xe^{x^2}}{1+e^{2x^2}}dx$$

Now, the problem reduces to proving

$$\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}dx \leq \int_a^b \frac{xe^{x^2}}{1+e^{2x^2}}dx$$

for which it suffices to prove

$$\frac{\sqrt{e^{x^2}}}{e^{3x^2}+1} \leq \frac{1}{1+e^{2x^2}}$$

Since the denominators are clearly positive, we can cross multiply

(1) $$e^{\frac{x^2}{2}} + e^{\frac{5x^2}{2}} \leq e^{3x^2} + 1$$

To prove this statement, we use Karamata Inequality.
Since sequence $(3,0) \succ (\frac{5}{2}, \frac{1}{2})$, and $f(a) = e^{ax^2}$ is convex whenever $a$ is positive, thus by Karamat inequality,

$$f(3) + f(0) \geq f\Big(\frac{5}{2}\Big) + f\Big(\frac{1}{2}\Big)$$

□

*Solution 5 by David A. Huckaby, Angelo State University, San Angelo, TX.*
Both sides of the inequality are 0 when $a = b$, so we assume $a < b$. The left-hand side of the inequality is

$$\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}dx = \int_a^b \frac{x(e^{x^2})^{\frac{3}{2}}}{e^{3x^2}+1}dx = \int_a^b \frac{x(e^{(\sqrt{3}x)^2})^{\frac{1}{2}}}{e^{(\sqrt{3}x)^2}+1}dx$$

Let $u = e^{(\sqrt{3}x)^2}$, so that $x = \sqrt{\frac{1}{3}\ln u}$. Then $du = e^{(\sqrt{3}x)^2}\cdot 2(\sqrt{3}x)\cdot\sqrt{3}dx = 6\sqrt{\frac{1}{3}\ln u}\cdot udx$, so that $dx = \frac{du}{6u\sqrt{\frac{1}{3}\ln u}}$. The integral is then

$$\int_{e^{(\sqrt{3}a)^2}}^{e^{(\sqrt{3}b)^2}} \frac{\sqrt{\frac{1}{3}\ln u}u^{\frac{1}{2}}}{u+1}\cdot\frac{du}{6u\sqrt{\frac{1}{3}\ln u}} = \frac{1}{6}\int_{e^{(\sqrt{3}a)^2}}^{e^{(\sqrt{3}b)^2}} \frac{du}{u^{\frac{1}{2}}(u+1)} =$$

$$= \frac{1}{6}\int_{e^{(\sqrt{3}a)^2}}^{e^{(\sqrt{3}b)^2}} \frac{u^{-\frac{1}{2}}du}{(u^{\frac{1}{2}})^2+1}$$

$$= \frac{1}{6}\Big[2\tan^{-1}(u^{\frac{1}{2}})\Big]_{e^{(\sqrt{3}a)^2}}^{e^{(\sqrt{3}b)^2}} = \frac{1}{3}\Big[\tan^{-1}(e^{(\sqrt{3}b)^2})^{\frac{1}{2}} - \tan^{-1}(e^{(\sqrt{3}a)^2})^{\frac{1}{2}}\Big]$$

$$= \frac{1}{3}(\tan^{-1}e^{\frac{3}{2}b^2} - \tan^{-1}e^{\frac{3}{2}a^2}).$$

Now from the identity $\tan^{-1}x - \tan^{-1}y = \frac{x-y}{1+xy}$, the right-hand side of the original inequality is

$$\frac{1}{2}\tan^{-1}\Big(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\Big) = \frac{1}{2}(\tan^{-1}e^{b^2} - \tan^{-1}e^{a^2}).$$

So the original inequality is

$$\frac{1}{3}(\tan^{-1}e^{\frac{3}{2}b^2} - \tan^{-1}e^{\frac{3}{2}a^2} \le \frac{1}{2}(\tan^{-1}e^{b^2} - \tan^{-1}e^{a^2}),$$

□

*Solution 6 by Michel Bataille, Rouen, France.*
Let $I$ be the integral on the left. The change of variables $x = \sqrt{\ln u}$ (so that $e^{x^2} = u$ and $dx = \frac{du}{2u\sqrt{\ln u}}$) leads to

$$I = \frac{1}{2}\int_{e^{a^2}}^{e^{b^2}} \frac{\sqrt{u}}{u^3+1}du = \frac{1}{2}\int_{e^{a^2}}^{e^{b^2}}\Big(\frac{1}{1+u^2} - \frac{((\sqrt{u})^5-1)(\sqrt{u}-1)}{(1+u^2)(1+u^3)}\Big)du.$$

Since $\sqrt{u}\ge 1$ when $u > 1$, hence when $e^{a^2}\le u\le e^{b^2}$, we see that

$$\frac{1}{1+u^2} - \frac{((\sqrt{u})^5-1)(\sqrt{u}-1)}{(1+u^2)(1+u^3)} \le \frac{1}{1+u^2}$$

and it follows that

$$I \le \frac{1}{2}\int_{e^{a^2}}^{e^{b^2}}\frac{du}{1+u^2} = \frac{1}{2}(\tan^{-1}(e^{b^2} - \tan^{-1}(e^{a^2})) =$$

$$= \frac{1}{2}\tan^{-1}\Big(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2}e^{b^2}}\Big) = \frac{1}{2}\tan^{-1}\Big(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2}+b^2}\Big).$$

□

*Solution 7 by proposer.*
For $x > 0 : (x-1)^2(x^4+x^3+x^2+x+1)\ge 0$

$$(x-1)(x^5-1)\ge 0 \Rightarrow x^5(x-1)-(x-1)\ge 0$$

$$x^6-x^5+1-x\ge 0 \Rightarrow (x^4+2x^2+1)x \le x^6+2x^3+1$$

$$(x^2+1)^2x \le (x^3+1)^2$$

$$(x^2+1)\sqrt{x} \le x^3+1 \Rightarrow \frac{\sqrt{x}}{x^3+1} \le \frac{1}{x^2+1} \tag{1}$$

Replacing $x$ in (1) with $e^{x^2}$ and multiplying with $2xe^{x^2}$:

$$\frac{2xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1} \le \frac{2xe^{x^2}}{e^{2x^2}+1}$$

$$\int_a^b \frac{2xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}dx \le \int_a^b \frac{2xe^{x^2}}{e^{2x^2}+1}dx =$$

$$= \tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})$$

$$\int_a^b \frac{xe^{x^2}\sqrt{e^{x^2}}}{e^{3x^2}+1}dx \le \frac{1}{2}(\tan^{-1}(e^{b^2}) - \tan^{-1}(e^{a^2})) =$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{e^{b^2}-e^{a^2}}{1+e^{a^2+b^2}}\right)$$

□

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