5810

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5810. If $x, y, z \in (0,1)$ and $xy + yz + zx = 1$, then show that:

$$(x^4 + y^4)(y^4 + z^4)(z^4 + x^4) \geq \frac{8}{729}$$

Daniel Sitaru

Solution 1 by Songen Tang (undergraduate student) and the Eagle Problem Solvers, Georgia, Southern University, Savannah, GA and Statesboro, GA.
We begin by proving the following lemmas.
Lemma 1: If x, y and z are real numbers, then

$$(x^4 + y^4)(y^4 + z^4)(z^4 + x^4) \geq \frac{8}{9}(x^4 + y^4 + z^4)(x^4y^4 + y^4z^4 + z^4x^4).$$

Proof.

$$\begin{aligned}
&9(x^4 + y^4)(y^4 + z^4)(z^4 + x^4) - 8(x^4 + y^4 + z^4)(x^4y^4 + y^4z^4 + z^4x^4)\\
&= 9(x^4y^4 + y^4z^4 + z^4x^4 + y^8)(z^4 + x^4) - 8(x^4 + y^4 + z^4)(x^4y^4 + y^4z^4 + z^4x^4)\\
&= 9y^8(z^4 + x^4) + (x^4 + y^4 + z^4)(z^4 + x^4) - 8y^4(x^4y^4 + y^4z^4 + z^4x^4)\\
&= y^8(z^4 + x^4) + (x^4 + y^4 + z^4)(z^4 + x^4) - 8x^4y^4z^4\\
&= (x^4 + y^4)(y^4 + z^4)(z^4 + x^4) - 8x^4y^4z^4\\
&\geq (2x^2y^2)(2y^2z^2)(2z^2x^2) - 8x^4y^4z^4\\
&= 0.
\end{aligned}$$

□

Lemma 2: If a, b, and c are real numbers, then $a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3}$.

Proof.
$3(a^2+b^2+c^2)-(a+b+c)^2 = 2(a^2+b^2+c^2)-2(ab+bc+ca) = (a-b)^2+(b-c)^2+(c-a)^2 \geq 0$
From Lemma 1, using x^2, y^2, and z^2 for a, b, and c, we have

$$x^4+y^4+z^4 \geq \frac{(x^2 + y^2 + z^2)^2}{3} = \frac{1}{3}\Big(\frac{x^2 + y^2}{2}+\frac{y^2 + z^2}{2}+\frac{z^2 + x^2}{2}\Big)^2 \geq \frac{1}{3}(xy+yz+zx)^2 = \frac{1}{3}.$$

Meanwhile, using Lemma 1 twice more, we get

$$x^4y^4+y^4z^4+z^4x^4 \geq \frac{(x^2y^2 + y^2z^2 + z^2x^2)^2}{3} \geq \frac{1}{3}\Big(\frac{(xy + yz + zx)^2}{3}\Big)^2 = \frac{1}{3}\cdot\frac{1}{9} = \frac{1}{27}.$$

Finally, using Lemma 2, we have

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geq \frac{8}{9}(x^4+y^4+z^4)(x^4y^4+y^4z^4+z^4x^4) \geq \frac{8}{9}\cdot\frac{1}{3}\cdot\frac{1}{27} = \frac{8}{729}.$$

□

□

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.
We apply the Power Mean inequality (P) and Maclaurin's inequality (M):

$$\left(\frac{x^4+y^4}{2}\right)^{\frac{1}{4}} \overset{\text{P}}{\geq} \frac{x+y}{2} \overset{\text{M}}{\geq} \sqrt{\frac{xy+yz+zx}{3}} \Rightarrow x^4+y^4 \geq 2\left(\frac{xy+yz+zx}{3}\right)^2 = \frac{2}{9}.$$

It follows that

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geq \left(\frac{2}{9}\right)^3 = \frac{8}{729}.$$

We note that only $x, y, z > 0$ is required and that equality holds if and only if $x = y = z = \frac{1}{\sqrt{3}}$. □

Solution 3 by Michel Bataille, Rouen, France.
For any real numbers a, b, we have $2(a^2+b^2) - (a+b)^2 = (a-b)^2 \geq 0$, hence $a^4 + b^4 \geq \frac{(a^2+b^2)^2}{2}$.
It follows that the required inequality will certainly hold if

$$(x^2+y^2)(y^2+z^2)(z^2+x^2) \geq \frac{8}{27}.$$

Again we have $x^2 + y^2 \geq \frac{(x+y)^2}{2}$, etc, hence the latter will hold if

Now, from

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) \geq 3(xy+yz+zx) = 3$$

and

$$1 = xy+yz+zx \geq 3\sqrt[3]{xy \cdot yz \cdot zx} = 3\sqrt[3]{(xyz)^2}$$

we obtain $x+y+z \geq \sqrt{3}$ and $xyz \leq \frac{1}{3\sqrt{3}}$. We deduce that

$$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz = x+y+z-xyz \geq \sqrt{3} - \frac{1}{3\sqrt{3}} = \frac{8}{3\sqrt{3}}.$$

Thus, (1) holds and the proof is complete. □

Solution 4 by Moti Levy, Rehovot, Israel.
For $u, v \geq 0$,

$$\left(\frac{u^4+v^4}{2}\right)^{\frac{1}{4}} \geq \frac{u+v}{2} \Rightarrow u^4+v^4 \geq \frac{(u+v)^4}{8}.$$

Applying this to the pairs $(x, y), (y, z), (z, x)$ and multiplying yields

$$(x^4+y^4)(y^4+z^4)(z^4+x^4) \geq \frac{((x+y)(y+z)(z+x))^4}{512}$$

Thus it suffices to show

(1) $$(x+y)(y+z)(z+x) \geq \frac{8}{3\sqrt{3}}$$

$$(x+y)(y+z)(z+x) = (x+y+z)(xy+yz+zx) - xyz.$$

Write $p = x+y+z$ and $r = xyz$, and because $xy+yz+zx = 1$, then:

$$(x+y)(y+z)(z+x) = p - r$$

Let $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$ so that $ab+bc+ca = 3$ and

$$p - r := \frac{1}{\sqrt{3}}\left(a+b+c-\frac{abc}{3}\right).$$

We claim that

$$a+b+c-\frac{abc}{3}\geq\frac{8}{3} \tag{2}$$

Indeed, from $ab+bc+ca=3$ we have

$$a^2+b^2+c^2\geq ab+bc+ca=3,$$

whence

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\geq 3+6=9\Rightarrow$$

$$a+b+c\geq 3 \tag{4}$$

Also, by AM-GM applied to ab, bc, ca,

$$1=\frac{ab+bc+ca}{3}\geq\sqrt[3]{(ab)(bc)(ca)}=(abc)^{\frac{2}{3}}$$

so

$$abc\leq 1 \tag{4}$$

Combining (3) and (4) gives,

$$p-r\geq\frac{1}{\sqrt{3}}\Big(3-\frac{1}{3}\Big)=\frac{8}{3\sqrt{3}}$$

Finally,

$$(x^4+y^4)(y^4+z^4)(z^4+x^4)\geq\frac{(\frac{8}{3\sqrt{3}})^4}{512}=\frac{8}{729}$$

□

Solution 5 by proposer.
First, we prove by induction that:

$$(x+y)^n\leq 2^{n-1}(x^n+y^n) \tag{1}$$

For $n=1\Rightarrow x+y\leq x+y$ (true)
$P(n):(x+y)^n\leq 2^{n-1}(x^n+y^n)$ (suppose true)
$P(n+1):(x+y)^{n+1}\leq 2^n(x^{n+1}+y^{n+1})$ (to prove)

$$(x+y)^{n+1}=(x+y)(x+y)^n\overset{P(n)}{\leq}(x+y)\cdot 2^{n-1}(x^n+y^n)\leq 2^n(x^{n+1}+y^{n+1})\Leftrightarrow$$
$$(x+y)(x^n+y^n)\leq 2(x^{n+1}+y^{n+1})\Leftrightarrow$$
$$\Leftrightarrow 2x^{n+1}+2y^{n+1}-x^{n+1}-y^{n+1}-x^ny-xy^n\geq 0$$
$$\Leftrightarrow x^{n+1}-x^ny+y^{n+1}-xy^n\geq 0$$
$$\Leftrightarrow x^n(x-y)-y^n(x-y)\geq 0$$
$$\Leftrightarrow (x-y)(x^n-y^n)\geq 0$$
$$\Leftrightarrow (x-y)^2(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots+y^{n-1})\geq 0$$
$$P(n)\rightarrow P(n+1)$$
$$x,y,z\in(0,1);xy+yz+zx=1\Rightarrow$$
$$\Rightarrow(\exists)\alpha,\beta,\gamma\in\Big(0,\frac{\pi}{4}\Big);x=\tan\alpha,y=\tan\beta,z=\tan\gamma$$
$$x(y+z)=1-yz\Rightarrow\frac{1-yz}{y+z}$$
$$\tan\alpha=\frac{1-\tan\beta\tan\gamma}{\tan\beta+\tan\gamma}\Rightarrow\tan\alpha=\cot(\beta+\gamma)$$

$$\Rightarrow \tan\alpha = \tan\Big(\frac{\pi}{2} - (\beta+\gamma)\Big) \Rightarrow \alpha = \frac{\pi}{2} - (\beta+\gamma)$$
$$\alpha+\beta+\gamma = \frac{\pi}{2} \Rightarrow 2\alpha+2\beta+2\gamma = \pi$$
$$\alpha = \frac{A}{2}; \beta = \frac{B}{2}; \gamma = \frac{C}{2}$$
$$x = \tan\frac{A}{2}; y = \tan\frac{B}{2}; z = \tan\frac{C}{2}$$

By (1) : $x^4 + y^4 \geq \frac{(x+y)^4}{2^{4-1}} = \frac{(x+y)^4}{8}$

$$\prod_{cyc}(x^4+y^4) \geq \prod_{cyc}\frac{(x+y)^4}{8} = \frac{1}{8^3}\prod_{cyc}(x+y)^4 =$$
$$= \frac{1}{2^9}\cdot\prod_{cyc}\Big(\tan\frac{A}{2}+\tan\frac{B}{2}\Big)^4 =$$
$$= \frac{1}{2^9}\cdot\left(\prod_{cyc}\left(\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} + \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\right)\right)^4 =$$
$$= \frac{1}{2^9}\left(\prod_{cyc}\sqrt{\frac{s-c}{s}}\left(\sqrt{\frac{s-b}{s-a}} + \sqrt{\frac{s-a}{s-b}}\right)\right)^4 =$$
$$= \frac{1}{2^9}\left(\prod_{cyc}\frac{\sqrt{s-c}}{\sqrt{s}}\cdot\frac{s-b+s-a}{\sqrt{(s-a)(s-b)}}\right)^4 =$$
$$= \frac{1}{2^9}\left(\prod_{cyc}\frac{c}{\sqrt{s(s-a)}}\right)^4 =$$
$$= \frac{1}{2^9}\Big(\frac{abc}{s\sqrt{s(s-a)(s-b)(s-c)}}\Big)^4 =$$
$$= \frac{1}{2^9}\Big(\frac{4RF}{sF}\Big)^4 \overset{\text{MITRINOVIC}}{\geq}$$
$$\geq \frac{1}{2^9}\cdot\Big(\frac{4\cdot\frac{4s}{3\sqrt{3}}}{s}\Big)^4 = \frac{1}{2^9}\cdot 2^{12}\cdot\frac{1}{81\cdot 9} = \frac{8}{729}$$

Equality holds for $A = B = C = \frac{\pi}{3} \Rightarrow x = y = z = \frac{\sqrt{3}}{3}$. □

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