ROMANIAN MATHEMATICAL MAGAZINE

NEW IDENTITIES AND INEQUALITIES IN TRIANGLE-(II)

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We consider ABC triangle , and: n_a-cevian of Nagel from A; g_a-cevian of Gergonne from A; p_a-cevian of Spieker from A ; p= $\frac{1}{2}$(a+b+c)

$$\mathbf{2r_ah_a\left(\frac{R}{r}-1\right)=n_a^2+r_a^2} \text{ (and analogous)(1)[1]}$$

$$\mathbf{\frac{r_a}{r}=\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}} \text{ (and analogous)(2)[2]}$$

$$\mathbf{p^2=n_a^2+2r_ah_a} \text{ (and analogous)(3)[1]}$$

$$\mathbf{n_a\geq p_a\sqrt{\frac{l_a}{g_a}}} \text{ (and analogous)(4)[3]}$$

$$2\frac{r_a}{r}h_a(R-r)=n_a^2+r_a^2 \rightarrow 2h_a\frac{n_a(R-r)}{n_a-\sqrt{4r^2+(b-c)^2}}=n_a^2+r_a^2$$

$$\mathbf{2h_a\frac{(R-r)}{n_a-\sqrt{4r^2+(b-c)^2}}=n_a+\frac{r_a^2}{n_a}} \text{ (and analogous)(5)}$$

From (5) after summation :

$$\mathbf{2(R-r)\sum\frac{h_a}{n_a-\sqrt{4r^2+(b-c)^2}}=n_a+n_b+n_c+\frac{r_a^2}{n_a}+\frac{r_b^2}{n_b}+\frac{r_c^2}{n_c}} \text{ (6)}$$

From (4) and (6) →

$$\mathbf{2h_a\frac{(R-r)}{p_a\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\geq n_a+\frac{r_a^2}{n_a}} \text{ (7)}$$

From(7) after summation:

$$\mathbf{2(R-r)\sum\frac{h_a}{p_a\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\geq n_a+n_b+n_c+\frac{r_a^2}{n_a}+\frac{r_b^2}{n_b}+\frac{r_c^2}{n_c}} \text{ (8)}$$

From Bergstrom :

$$\mathbf{\frac{r_a^2}{n_a}+\frac{r_b^2}{n_b}+\frac{r_c^2}{n_c}\geq\frac{(r_a+r_b+r_c)^2}{n_a+n_b+n_c}} \text{ (9)}$$

From (8) and (9) :

$$2(R-r)\sum\frac{h_a}{p_a\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\geq n_a+n_b+n_c+\frac{(r_a+r_b+r_c)^2}{n_a+n_b+n_c} \quad (10)$$

From (4) :$n_a^2 \geq n_a p_a \sqrt{\frac{l_a}{g_a}}$ and (3) :

$p^2 \geq n_a p_a \sqrt{\frac{l_a}{g_a}} + 2r_a h_a$ or :

$$\mathbf{p} \geq \sqrt{\mathbf{n_a p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \mathbf{2r_a h_a}}\ \textbf{(and analogous) (11)}$$

From $\frac{p}{h_a} = \frac{a}{2r}$ (and analogous) and (11):

$$\frac{\mathbf{a}}{\mathbf{2r}} \geq \sqrt{\frac{\mathbf{n_a p_a}}{\mathbf{h_a^2}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \frac{\mathbf{2r_a}}{\mathbf{h_a}}}\ \textbf{(and analogous) (12)}$$

From (12) after summation:

$$\frac{\mathbf{p}}{\mathbf{r}} \geq \sum \sqrt{\frac{\mathbf{n_a p_a}}{\mathbf{h_a^2}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \frac{\mathbf{2r_a}}{\mathbf{h_a}}}\ \textbf{(13)}$$

From $r_a = \frac{S}{p-a}$ (and analogous) and S= pr, $\frac{r_a}{r} = \frac{p}{p-a} = \frac{p^2}{p(p-a)}$ (and analogous);

Also is well-known that :$r_b r_c = p(p-a)$ (and analogous);

$\frac{r_a}{r} = \frac{p}{p-a} = \frac{p^2}{p(p-a)} = \frac{2p^2}{2r_b r_c}$ and from (3):

$\frac{r_a}{r} = \frac{n_c^2 + 2r_c h_c + n_b^2 + 2r_b h_b}{2r_b r_c} \geq \frac{2(n_b n_c + r_c h_c + r_c h_c)}{2r_b r_c}$ (and analogous)

$$\frac{\mathbf{r_a}}{\mathbf{r}} \geq \frac{\mathbf{n_b n_c + r_c h_c + r_b h_b}}{\mathbf{r_b r_c}} = \frac{\mathbf{n_b n_c}}{\mathbf{r_b r_c}} + \frac{\mathbf{h_c}}{\mathbf{r_b}} + \frac{\mathbf{h_b}}{\mathbf{r_c}}\ \textbf{(and analogous) (14)}$$

From (14) and (2) :

$$\frac{\mathbf{n_a}}{\mathbf{n_a} - \sqrt{\mathbf{4r^2 + (b-c)^2}}} \geq \frac{\mathbf{n_b n_c}}{\mathbf{r_b r_c}} + \frac{\mathbf{h_c}}{\mathbf{r_b}} + \frac{\mathbf{h_b}}{\mathbf{r_c}}\ \textbf{(and analogous) (15)}$$

From $r_a r_b r_c = p^2 r$ (well-known) and (14):

$$\left(\frac{\mathbf{p}}{\mathbf{r}}\right)^{\mathbf{2}} \geq \prod \left(\frac{\mathbf{n_b n_c}}{\mathbf{r_b r_c}} + \frac{\mathbf{h_c}}{\mathbf{r_b}} + \frac{\mathbf{h_b}}{\mathbf{r_c}}\right)\ \textbf{(16)}$$

From $\frac{r_a}{r} = \frac{p}{p-a} = \frac{p^2}{p(p-a)} = \frac{p^2}{r_b r_c}$ and (3) :

$\frac{r_a}{r} = \frac{n_b^2}{r_b r_c} + \frac{2r_b h_b}{r_b r_c} = \frac{n_b^2}{r_b r_c} + \frac{2h_b}{r_c}$ (and analogous). From $r_a + r_b + r_c = 4R + r$ (well-known) ;

$r_b r_c + r_a r_c + r_a r_b = p^2$ (well-known) and using Bergstrom Inequality :

$$1+\frac{4R}{r} \geq \left(\frac{n_a+n_b+n_c}{p}\right)^2 + 2\left(\frac{h_a}{r_b} + \frac{h_b}{r_c} + \frac{h_c}{r_a}\right) (17)$$

We consider :p−b and p−c. For AM-GM : p−b+ p−c≥$2\sqrt{(p-c)(p-b)}$

$\sqrt{(p-c)(p-b)} = \sqrt{rr_a}$ (and analogous) (well-known, p−b+ p−c=a

a≥$2\sqrt{rr_a}$→$\frac{a}{2r} \geq \sqrt{\frac{r_a}{r}}$ and $\frac{r_a}{r} = \frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$ (and analogous):

$$\frac{a}{2r} \geq \sqrt{\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}} \text{ (and analogs)(18)}$$

From (18) and (4) :

$$\frac{a}{2r} \geq \sqrt{\frac{p_a\sqrt{\frac{l_a}{g_a}}}{n_a-\sqrt{4r^2+(b-c)^2}}} \text{ (and analogous) (19)}$$

From (18) after summation :

$$\frac{p}{r} \geq \sum \sqrt{\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}} \text{ (20)}$$

From (19) after summation :

$$\frac{p}{r} \geq \sum \sqrt{\frac{p_a\sqrt{\frac{l_a}{g_a}}}{n_a-\sqrt{4r^2+(b-c)^2}}} \text{ (21)}$$

From $\frac{a}{2r} \geq \sqrt{\frac{r_a}{r}}$ and (14) :

$$\frac{a}{2r} \geq \sqrt{\frac{n_b n_c}{r_b r_c} + \frac{h_c}{r_b} + \frac{h_b}{r_c}} \text{ (and analogous) (22)}$$

From (22) after summation :

$$\frac{p}{r} \geq \sum \sqrt{\frac{n_b n_c}{r_b r_c} + \frac{h_c}{r_b} + \frac{h_b}{r_c}} \text{ (23)}$$

From $\frac{r_a}{r} = \frac{n_b^2}{r_b r_c} + \frac{2r_b h_b}{r_b r_c} = \frac{n_b^2}{r_b r_c} + \frac{2h_b}{r_c}$ (and analogous) and $\frac{a}{2r} \geq \sqrt{\frac{r_a}{r}}$ we obtain :

$$\frac{a}{2r} \geq \sqrt{\frac{n_b^2}{r_b r_c} + \frac{2h_b}{r_c}} \text{ (and analogous) (24)}$$

From (24) and (4):

$$\frac{a}{2r} \geq \sqrt{\frac{\left(p_b\sqrt{\frac{l_b}{g_b}}\right)^2}{r_b r_c} + \frac{2h_b}{r_c}} \textbf{ (and analogous) (25)}$$

From (25) after summation:

$$\frac{p}{r} \geq \sum \sqrt{\frac{\left(p_b\sqrt{\frac{l_b}{g_b}}\right)^2}{r_b r_c} + \frac{2h_b}{r_c}} \textbf{ (26)}$$

From (2) and $n_a^2 \geq n_a p_a\sqrt{\frac{l_a}{g_a}}$:

$$\frac{r_a}{r} \geq \frac{\sqrt{n_a p_a\sqrt{\frac{l_a}{g_a}}}}{n_a - \sqrt{4r^2+(b-c)^2}} \textbf{ (and analogous) (27)}$$

From (27) and $r_a + r_b + r_c = 4R + r$:

$$1+\frac{4R}{r} \geq \sum \frac{\sqrt{n_a p_a\sqrt{\frac{l_a}{g_a}}}}{n_a - \sqrt{4r^2+(b-c)^2}} \textbf{ (28)}$$

Is well-known that :$AI=\frac{r}{\sin\frac{A}{2}}$ (and analogous); $bc=p(p-a)+(p-c)(p-b)$(and analogous);

$(p-c)(p-b) = rr_a$(and analogous); $p(p-a)= r_b r_c$ (and analogous);

$\cos\frac{A}{2} = \sqrt{\frac{r_b r_c}{bc}}$ (and analogous); $\sin\frac{A}{2} = \sqrt{\frac{rr_a}{bc}}$ (and analogous) ; S=pr ; $rr_a r_b r_c=S^2$;

Now $\frac{AI}{r}=\sqrt{\frac{bc}{rr_a}}=\frac{1}{\sin\frac{A}{2}}=\sqrt{1+\frac{r_b r_c}{rr_a}}$; After simple simplifications: $\cot\frac{A}{2}=\frac{p}{r_a}$ (and analogous);

Is also well-known : $r_a(p-a)=S$; $r_a(p-a)= pr \rightarrow \frac{p}{r_a}=\frac{p-a}{r}$

$\rightarrow \frac{AI}{r}=\sqrt{\frac{bc}{rr_a}}=\frac{1}{\sin\frac{A}{2}}=\sqrt{1+\frac{r_b r_c}{rr_a}}=\sqrt{1+\left(\frac{p}{r_a}\right)^2}=\sqrt{1+\left(\frac{p-a}{r}\right)^2}$ (and analogous);

$$\rightarrow AI=\sqrt{r^2+(p-a)^2} \textbf{ (and analogous)(29)}$$

From :$\frac{r_a}{\sin\frac{A}{2}}=\sqrt{p^2+r_a^2}$ (and analogous) and (3) :

$$\frac{r_a}{\sin\frac{A}{2}} = \sqrt{r_a^2 + n_a^2 + 2r_a h_a} \textbf{ (and analogous) (30)}$$

Because $n_a \geq h_a$ and (30):

$$\mathbf{n_a + r_a \geq \frac{r_a}{\sin\frac{A}{2}}} \textbf{ (and analogous) (31)}$$

From (31) and $\frac{AI}{r} = \frac{1}{\sin\frac{A}{2}} \rightarrow$

$$\mathbf{1+\frac{n_a}{r_a} \geq \frac{AI}{r}} \textbf{ (32)}$$

From (31) and (2) : $n_a + r_a \geq \frac{n_a}{n_a - \sqrt{4r^2+(b-c)^2}} AI$

$$\mathbf{1+\frac{r_a}{n_a} \geq \frac{AI}{n_a - \sqrt{4r^2+(b-c)^2}}} \textbf{ (and analogous) (33)}$$

From (33) after summation:

$$\mathbf{3+\frac{r_a}{n_a} + \frac{r_a}{n_a} + \frac{r_a}{n_a} \geq \sum \frac{AI}{n_a - \sqrt{4r^2+(b-c)^2}}} \textbf{ (34)}$$

From (2) : $\frac{n_a}{r_a} = \frac{n_a - \sqrt{4r^2+(b-c)^2}}{r}$ (and analogous) and (32):

$$\mathbf{1+\frac{n_a - \sqrt{4r^2+(b-c)^2}}{r} \geq \frac{AI}{r}} \textbf{ (and analogous) (35)}$$

From (35) :

$$\mathbf{r+n_a \geq AI+\sqrt{4r^2 + (b-c)^2}} \textbf{ and analogous) (36)}$$

From (36) after summation:

$$\mathbf{3r+n_a + n_b + n_c \geq AI + BI + CI + \sum \sqrt{4r^2 + (b-c)^2}} \textbf{ (37)}$$

From (11) and (30):

$$\mathbf{\frac{r_a}{\sin\frac{A}{2}} \geq \sqrt{r_a^2 + n_a p_a \sqrt{\frac{l_a}{g_a}} + 2r_a h_a}} \textbf{ (and analogous) (38)}$$

After summation:

$$\mathbf{\sum \frac{r_a}{\sin\frac{A}{2}} \geq \sum \sqrt{r_a^2 + n_a p_a \sqrt{\frac{l_a}{g_a}} + 2r_a h_a}} \textbf{ (39)}$$

From (38) :

$$\mathbf{\frac{AI}{r} \geq \sqrt{1 + \frac{n_a p_a \sqrt{\frac{l_a}{g_a}}}{r_a^2} + 2\frac{h_a}{r_a}}} \textbf{ (and analogous) (40)}$$

From (40) after summation:

$$\frac{\mathbf{AI+BI+CI}}{\mathbf{r}} \geq \sum \sqrt{\mathbf{1} + \frac{\mathbf{n_a p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}}{\mathbf{r_a^2}} + \mathbf{2}\frac{\mathbf{h_a}}{\mathbf{r_a}}} \ \mathbf{(41)}$$

From (30) : $\frac{AI}{r} = \sqrt{1 + \frac{n_a^2}{r_a^2} + 2\frac{h_a}{r_a}}$ and $\frac{AI}{r} = \sqrt{1 + \left(\frac{p-a}{r}\right)^2}$:

$$\frac{\mathbf{n_a^2}}{\mathbf{r_a^2}} + \mathbf{2}\frac{\mathbf{h_a}}{\mathbf{r_a}} = \left(\frac{\mathbf{p-a}}{\mathbf{r}}\right)^2 \ (\textbf{and analogous})(\mathbf{42})$$

From $\frac{n_a}{r_a} = \frac{n_a - \sqrt{4r^2+(b-c)^2}}{r}$ (and analogous) and (42):

$$\mathbf{2}\frac{\mathbf{h_a}}{\mathbf{r_a}} = \frac{(\mathbf{p-a})^2 - \left(\mathbf{n_a} - \sqrt{\mathbf{4r^2+(b-c)^2}}\right)^2}{\mathbf{r^2}} \ \textbf{(and analogous) (43)}$$

From (43) and AI=$\sqrt{r^2 + (p-a)^2}$ (and analogous) :

$$2\frac{h_a}{r_a} = \frac{AI^2 - \left(n_a - \sqrt{4r^2+(b-c)^2}\right)^2 - r^2}{r^2}$$

$$\mathbf{1+2}\frac{\mathbf{h_a}}{\mathbf{r_a}} = \frac{\mathbf{AI^2} - \left(\mathbf{n_a} - \sqrt{\mathbf{4r^2+(b-c)^2}}\right)^2}{\mathbf{r^2}} \ \textbf{(and analogous)(44)}$$

From (44) : AI $> n_a - \sqrt{4r^2 + (b-c)^2}$

$$\mathbf{AI} + \sqrt{\mathbf{4r^2 + (b-c)^2}} > \mathbf{n_a} \ (\textbf{and analogous})(\mathbf{45})$$

From (45) and (36) :

$$\mathbf{r+n_a} \geq \mathbf{AI} + \sqrt{\mathbf{4r^2 + (b-c)^2}} > \mathbf{n_a} \ (\textbf{and analogous})(\mathbf{46})$$

From (43) :

$$\frac{\mathbf{r_a}}{\mathbf{r}} = \mathbf{2r}\frac{\mathbf{h_a}}{(\mathbf{p-a})^2 - \left(\mathbf{n_a} - \sqrt{\mathbf{4r^2+(b-c)^2}}\right)^2} \ \textbf{(and analogous) (47)}$$

From (47) after summation :

$$\frac{\mathbf{4R+r}}{\mathbf{2r^2}} = \sum \frac{\mathbf{h_a}}{(\mathbf{p-a})^2 - \left(\mathbf{n_a} - \sqrt{\mathbf{4r^2+(b-c)^2}}\right)^2} \ \mathbf{(48)}$$

From (47) and (2) :

$$\frac{n_a}{n_a - \sqrt{4r^2 + (b-c)^2}} = 2r\frac{h_a}{(p-a)^2 - \left(n_a - \sqrt{4r^2 + (b-c)^2}\right)^2}$$

$$\frac{\mathbf{n_a}}{\mathbf{h_a}}=\mathbf{2r}\frac{\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}}{\mathbf{(p-a)^2}-\left(\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}\right)^2}$$ **(and analogous) (49)**

From $\frac{p}{h_a}=\frac{a}{2r}$ (and analogous) $\rightarrow\frac{p-a}{h_a-2r}=\frac{a}{2r}$ because p>a; $h_a>2r$

2S=$h_a\,a=2pr=(a+b+c)r\rightarrow h_a=\left(1+\frac{b+c}{a}\right)r>2r$ because b+c>a (triangle inequality);

2p=a+b+c>2a→b+c>a(triangle inequality);

$h_a-2r=\left(1+\frac{b+c}{a}\right)r-2r=\left(\frac{b+c-a}{a}\right)r=\frac{2(a+b+c)-2a}{a}r=2\frac{(p-a)}{a}\,r$

$h_a-2r=2r\frac{(p-a)}{a}\rightarrow a(h_a-2r)=2r(p-a)\rightarrow\frac{p-a}{h_a-2r}=\frac{a}{2r}$

From : $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous)and (12) :

$$\frac{\mathbf{p-a}}{\mathbf{h_a-2r}}\geq\sqrt{\frac{\mathbf{n_a p_a}}{\mathbf{h_a^2}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}+\frac{\mathbf{2r_a}}{\mathbf{h_a}}}$$ **(and analogous)(50)**

From : $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous)and (18):

$$\frac{\mathbf{p-a}}{\mathbf{h_a-2r}}\geq\sqrt{\frac{\mathbf{n_a}}{\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}}}$$ **(and analogs)(51)**

From : $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous)and(19):

$$\frac{\mathbf{p-a}}{\mathbf{h_a-2r}}\geq\sqrt{\frac{\mathbf{p_a}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}}{\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}}}$$ **(and analogous) (52)**

From : $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous) and (22):

$$\frac{\mathbf{p-a}}{\mathbf{h_a-2r}}\geq\sqrt{\frac{\mathbf{n_b n_c}}{\mathbf{r_b r_c}}+\frac{\mathbf{h_c}}{\mathbf{r_b}}+\frac{\mathbf{h_b}}{\mathbf{r_c}}}$$ **(and analogous)(53)**

From : $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous) and (24):

$$\frac{\mathbf{p-a}}{\mathbf{h_a-2r}}\geq\sqrt{\frac{\mathbf{n_b^2}}{\mathbf{r_b r_c}}+\frac{\mathbf{2h_b}}{\mathbf{r_c}}}$$ **(and analogous)(54)**

From : $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous) and (25):

$$\frac{p-a}{h_a-2r} \geq \sqrt{\frac{\left(p_b\sqrt{\frac{l_b}{g_b}}\right)^2}{r_b r_c} + \frac{2h_b}{r_c}} \text{ (and analogous)(55)}$$

From : $\frac{p-a}{h_a-2r} = \frac{a}{2r}$ (and analogous) after summation:

$$\sum \frac{p-a}{h_a-2r} = \frac{p}{r} \quad (56)$$

From (56) and (13):

$$\sum \frac{p-a}{h_a-2r} \geq \sum \sqrt{\frac{n_a p_a}{h_a^2}\sqrt{\frac{l_a}{g_a}} + \frac{2r_a}{h_a}} \quad (57)$$

From (56) and (16):

$$\left(\sum \frac{p-a}{h_a-2r}\right)^2 \geq \prod \left(\frac{n_b n_c}{r_b r_c} + \frac{h_c}{r_b} + \frac{h_b}{r_c}\right) (58)$$

From (56) and (20):

$$\sum \frac{p-a}{h_a-2r} \geq \sum \sqrt{\frac{n_a}{n_a - \sqrt{4r^2+(b-c)^2}}} \quad (59)$$

From (56) and (21):

$$\sum \frac{p-a}{h_a-2r} \geq \sum \sqrt{\frac{p_a\sqrt{\frac{l_a}{g_a}}}{n_a - \sqrt{4r^2+(b-c)^2}}} \quad (60)$$

From (56) and (23):

$$\sum \frac{p-a}{h_a-2r} \geq \sum \sqrt{\frac{n_b n_c}{r_b r_c} + \frac{h_c}{r_b} + \frac{h_b}{r_c}} \quad (61)$$

From (56) and (26):

$$\sum \frac{p-a}{h_a-2r} \geq \sum \sqrt{\frac{\left(p_b\sqrt{\frac{l_b}{g_b}}\right)^2}{r_b r_c} + \frac{2h_b}{r_c}} \quad (62)$$

From $\frac{p-a}{h_a-2r} = \frac{a}{2r}$ (and analogous) and (12):

$$\frac{a(p-a)}{2r(h_a-2r)} \geq \frac{n_a p_a}{h_a^2}\sqrt{\frac{l_a}{g_a}} + \frac{2r_a}{h_a} \text{(and analogous)(63)}$$

From $\frac{p-a}{h_a-2r} = \frac{a}{2r}$ (and analogous) and (18):

$$\frac{a(p-a)}{2r(h_a-2r)} \ge \frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} \text{ (and analogous)}(64)$$

From $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous) and (19):

$$\frac{a(p-a)}{2r(h_a-2r)} \ge \frac{p_a\sqrt{\frac{l_a}{g_a}}}{n_a-\sqrt{4r^2+(b-c)^2}} \text{ (and analogous)(65)}$$

From $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous) and (22):

$$\frac{a(p-a)}{2r(h_a-2r)} \ge \frac{n_b n_c}{r_b r_c}+\frac{h_c}{r_b}+\frac{h_b}{r_c} \text{ (and analogous) (66)}$$

From $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous) and (24)

$$\frac{a(p-a)}{2r(h_a-2r)} \ge \frac{n_b^2}{r_b r_c}+\frac{2h_b}{r_c} \text{ (and analogous)}(67)$$

From $\frac{p-a}{h_a-2r}=\frac{a}{2r}$ (and analogous) and (25):

$$\frac{a(p-a)}{2r(h_a-2r)} \ge \frac{\left(p_b\sqrt{\frac{l_b}{g_b}}\right)^2}{r_b r_c}+\frac{2h_b}{r_c} \text{ (and analogous)}(68)$$

From $\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogous)[4] and $\frac{x}{y}=\frac{A}{B}$ (x,y,A,B>0)

A>x ;B>y $\rightarrow \frac{x}{y}=\frac{A-x}{B-y}$ obtain: $\frac{n_a}{h_a}=\frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}$

From $\frac{n_a}{h_a}=\frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}$ (and analogous) and (4):

$$\frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r} \ge \frac{p_a}{h_a}\sqrt{\frac{l_a}{g_a}} \text{ (and analogous) (69)}$$

$h_a-2r=\left(\frac{b+c-a}{a}\right)r=\frac{2(p-a)}{a}$; $h_a=\frac{2pr}{a} \rightarrow \frac{h_a}{h_a-2r}=\frac{2pr}{a}\left(\frac{a}{b+c-a}\right)\frac{1}{r}$

$\frac{h_a}{h_a-2r}=\frac{p}{p-a}=\frac{r_a}{r}$ (and analogous); From (69) and $\frac{h_a}{h_a-2r}=\frac{p}{p-a}=\frac{r_a}{r} \rightarrow \frac{n_a-\sqrt{4r^2+(b-c)^2}}{p_a}\sqrt{\frac{g_a}{l_a}} \ge \frac{r}{r_a}$

Is well-known that :$\sum\frac{1}{r_a}$=1 and from $\frac{n_a-\sqrt{4r^2+(b-c)^2}}{p_a}\sqrt{\frac{g_a}{l_a}} \ge \frac{r}{r_a}$ (and analogous) after summation:

$$\sum \frac{\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}}{\mathbf{p_a}}\sqrt{\frac{\mathbf{g_a}}{\mathbf{l_a}}} \geq \mathbf{1}\ \mathbf{(70)}$$

From (1): $2r_a h_a\left(\frac{R}{r}-1\right) = n_a^2 + r_a^2 \geq 2n_a r_a \to \frac{R}{r}-1 \geq \frac{n_a}{h_a}$

From $\frac{n_a}{h_a} = \frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}$ (and analogous) and $n_a^2 \geq n_a p_a\sqrt{\frac{l_a}{g_a}}$:

$\left(\frac{n_a}{h_a}\right)^2 \geq \frac{\left(n_a-\sqrt{4r^2+(b-c)^2}\right)p_a}{h_a(h_a-2r)}\sqrt{\frac{l_a}{g_a}}$ and $\frac{R}{r}-1 \geq \frac{n_a}{h_a}$:

$$\frac{\mathbf{R}}{\mathbf{r}} \geq \mathbf{1} + \sqrt{\frac{\left(\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}\right)\mathbf{p_a}}{\mathbf{h_a(h_a-2r)}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}}}\ \textbf{(and analogous) (71)}$$

From (12) and $\frac{n_a}{h_a} = \frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}$ (and analogous):

$$\frac{\mathbf{a}}{\mathbf{2r}} \geq \sqrt{\frac{\mathbf{p_a}\left(\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}\right)}{\mathbf{h_a(h_a-2r)}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \frac{\mathbf{2r_a}}{\mathbf{h_a}}}\ \textbf{(and analogous) (72)}$$

From (72) after summation:

$$\frac{\mathbf{p}}{\mathbf{r}} \geq \sum\sqrt{\frac{\mathbf{p_a}\left(\mathbf{n_a}-\sqrt{\mathbf{4r^2+(b-c)^2}}\right)}{\mathbf{h_a(h_a-2r)}}\sqrt{\frac{\mathbf{l_a}}{\mathbf{g_a}}} + \frac{\mathbf{2r_a}}{\mathbf{h_a}}}\ \mathbf{(73)}$$

From $r_b r_c + r_a r_c + r_a r_b = p^2$ (well-known) and $h_a = \frac{2r_b r_c}{r_b+r_c}$ (and analogous) (also well-known and easy to prove) and $p^2 = n_a^2 + 2r_a h_a$ (and analogous):

$r_b r_c + r_a r_c + r_a r_b = n_a^2 + 2r_a h_a \to n_a^2 = r_b r_c + r_a(r_b + r_c - 2h_a)$

$$n_a^2 = r_b r_c + r_a\left(r_b + r_c - \frac{4r_b r_c}{r_b+r_c}\right) = r_b r_c + \frac{r_a}{r_b + r_c}(r_b-r_c)^2$$

$$\mathbf{n_a^2 = r_b r_c + \frac{r_a}{r_b + r_c}(r_b-r_c)^2}\ \textbf{(and analogous)(74)}$$

From (74) and $n_a^2 \geq n_a p_a\sqrt{\frac{l_a}{g_a}} \to r_b r_c + \frac{r_a}{r_b+r_c}(r_b-r_c)^2 \geq n_a p_a\sqrt{\frac{l_a}{g_a}}$

$$\mathbf{\frac{r_a}{r_b + r_c}(r_b-r_c)^2 \geq n_a p_a\sqrt{\frac{l_a}{g_a}} - r_b r_c}\ \textbf{(and analogous)(75)}$$

From (75): $|r_b - r_c|\sqrt{\frac{r_a}{r_b+r_c}} \geq \sqrt{n_a p_a\sqrt{\frac{l_a}{g_a}} - r_b r_c}$ (and analogous) and after summation:

$$\sum |r_b - r_c|\sqrt{\frac{r_a}{r_b+r_c}} \geq \sum \sqrt{n_a p_a\sqrt{\frac{l_a}{g_a}} - r_b r_c} \quad (76)$$

From $|b-c| \geq n_a - g_a$ (and analogous)[5] and (2) :

$$\frac{r_a}{r} \geq \frac{n_a}{n_a - \sqrt{4r^2 + (n_a - g_a)^2}} \text{ (and analogous)(77)}$$

From (77) after summation:

$$1 + \frac{4R}{r} \geq \sum \frac{n_a}{n_a - \sqrt{4r^2 + (n_a - g_a)^2}} \quad (78)$$

From $|b-c| \geq n_a - g_a$ (and analogous)and (18):

$$\frac{a}{2r} \geq \sqrt{\frac{n_a}{n_a - \sqrt{4r^2 + (n_a - g_a)^2}}} \text{ (and analogs) (79)}$$

From $|b-c| \geq n_a - g_a$ (and analogous)and (19):

$$\frac{a}{2r} \geq \sqrt{\frac{p_a\sqrt{\frac{l_a}{g_a}}}{n_a - \sqrt{4r^2 + (n_a - g_a)^2}}} \text{ (and analogous) (80)}$$

From $|b-c| \geq n_a - g_a$ (and analogous)and (20):

$$\frac{p}{r} \geq \sum \sqrt{\frac{n_a}{n_a - \sqrt{4r^2 + (n_a - g_a)^2}}} \quad (81)$$

From $|b-c| \geq n_a - g_a$ (and analogous)and (21):

$$\frac{p}{r} \geq \sum \sqrt{\frac{p_a\sqrt{\frac{l_a}{g_a}}}{n_a - \sqrt{4r^2 + (n_a - g_a)^2}}} \quad (82)$$

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