SEQUENCES WITH THE RELATIONSHIP:

$$\sum_{k=1}^{n} p^{k-1} a_k \;=\; \prod_{k=1}^{n} a_k$$

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The relationship given in the title of the present paper hides a special, elegant, mysterious and very interesting sequence. It is worthy to deal with it. We may draft new open questions. I have been working on this subject for several years, and now I would like to present some features, by examining general and proper cases.

1. General definition and examinaton

1.1. We define the (a_n) sequence as follows: $a_1 = a \neq p$, $p > 0$, $a_2 = \dfrac{a}{a-p}$,

$a_{n+1} = \dfrac{a_n^2}{a_n^2 - pa_n + p}$, for $n \geq 2$. For $a = 0$ each term of the (a_n) sequence is 0, therefore we will deal with the cases when $a \neq 0$. If we have for p: $0 < p < 4$ then remains the only one interpretational condition. For $p \geq 4$ we should give some new conditions apart from the given one. We do not deal with this now.

1.2. We have valid the next context in the domain of interpretation of the sequence:

$a_1 + pa_2 + p^2 a_3 + ... + p^{n-1} a_n = a_1 \cdot a_2 \cdot a_3 \cdot ... \cdot a_n$ for any $n \geq 1$.

Proof: In case of n = 1 the equality is obvious. For n = 2 : $a_1 + pa_2 = a + \dfrac{pa}{a-p} = \dfrac{a^2}{a-p} =$

$a_1 \cdot a_2$. results from this $a_2 = \dfrac{a_1}{a_1 - p} \Leftrightarrow a_1 = \dfrac{pa_2}{a_2 - 1}$.

For any $n \geq 2$ we have the recursive context: $a_{n+1} = \dfrac{a_n^2}{a_n^2 - pa_n + p}$. Transforming this we will

get: $\dfrac{a_{n+1}}{a_{n+1} - 1} = \dfrac{a_n^2}{p(a_n - 1)}$ From this results, on the one hand: $a_n = p \cdot \dfrac{a_{n+1}}{a_n} \cdot \dfrac{a_n - 1}{a_{n+1} - 1}$, on the

other hand: $a_n = p \cdot \dfrac{a_{n+1}}{a_{n+1} - 1} - \dfrac{a_n}{a_n - 1}$. Writing this results for the n = 2 , 3 , . . . etc.

values, then multiply in the first case, or let us multiplied with the p , p^2 , p^3 , . . . etc

factors in the second case. After that we will add the terms taking into consideration that

$$\frac{pa_2}{a_2 - 1} = a_1 \ .$$

It follows that on the one hand: $a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n = p^n \cdot \dfrac{a_{n+1}}{a_{n+1} - 1}$,

on the other hand: $a_1 + pa_2 + p^2 a_3 + \ldots + p^{n-1} a_n = p^n \cdot \dfrac{a_{n+1}}{a_{n+1} - 1}$. From this we obtain the

above given relation.

1.3. For $0 < p < 4$, then based on $a_{n+1} - 1 = \dfrac{p(a_n - 1)}{a_n^2 - pa_n + p}$, $n \geq 2$ relation, if any term of the

sequence is greater than 1, then all the following terms will be greater than 1 too; if any term of the sequence is less than 1, then all the following terms will be less than 1 too. The result is valid for cases of $p \geq 4$ inside the domain of interpretation.

1.4. We can not draw a general conclusion either from the $a_{n+1} - p =$

$\dfrac{-(a_n - p)((p-1)a_n - p)}{a_n^2 - pa_n + p}$ relation, nor from the $a_{n+1} - a_n = \dfrac{-a_n(a_n - 1)(a_n - p)}{a_n^2 - pa_n + p}$ and $a_{n+2} -$

$a_n = \dfrac{-a_n(a_n - 1)(a_n - p)(a_n^2 - p(p-1)a_n + p^2)}{a_n^4 - p^2(a_n - 1)(a_n^2 - pa_n + p)}$, $n \geq 2$ relations. But later we can make use

of them in particular cases. If the sequence is convergent, then taking the limit of the recursive relation we find that the limit value can be only the 0, 1 or the p.

2. Examinaton of particular cases

2.1. Let it be p = 1. It is very interesting and elegant. We refound a simple classic case.

In this instance our sequence looks like: $a_1 = a \neq 1$, $a_2 = \dfrac{a}{a-1}$, $a_{n+1} = \dfrac{a_n^2}{a_n^2 - a_n + 1}$, for

any $n \geq 2$. Always defined, having all the terms positive numbers from the 3rd term. Is valid the next relation: $a_1 + a_2 + a_3 + \ldots + a_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$ for any $n \geq 1$. We have $a_{n+1} -$

$a_n = \dfrac{-a_n(a_n - 1)^2}{a_n^2 - a_n + 1} < 0$, which means that the sequence is severely decreasing. If $a > 1$,

then $a_2 > 1$ and the sequence has all terms greater than 1, is severely decreasing, convergent, the limit value being 1 . For $a < 1$, then a_2 can be negativ, or a value between 0 and 1 , but the a_3 value surely is less than 1, being a positive number and similarly the following terms of the sequence. In this case the sequence is severely decreasing, convergent, and the limit value is 0 .

The examination of this sequence was set repeatedly task on mathematics competitions like VII. NMMV Szabadka, 1998, ([3], [9]), Mathematical Olympic , local section, Szatmár country, 2005, [14], and it appeared in mathematical scientific reviews and mathematical collections of examples. It was edited by: Kovács Béla.

It is an open question the determaination of the general terms of the sequence. This was formulated repeatedly already, and experiments are proceeding about this direction: ([2], [5], [6], [7], [8], [10], [11]).

Comment: Introducing the $g(a) = \dfrac{a}{a-1}$ and $f(a) = \dfrac{a^2}{a^2-a+1}$ functions, the sequence can be written as follows: $a_1 = a \neq 1$, $a_2 = g(a)$ and

$a_n = \underbrace{f \circ f \circ \ldots \circ f}_{n-2}(g(a))$, $n \geq 3$, but this cannot be considered as the general member's closed form.

2.2. In the second case let us take p = 2. It is a very interesting and special case. Now the sequence has the following form: $a_1 = a \neq 2$, $a_2 = \dfrac{a}{a-2}$,

$a_{n+1} = \dfrac{a_n^2}{a_n^2 - 2a_n + 2}$, for each $n \geq 2$. Always defined, having all the terms positive numbers from the 3rd term. Is valid the next relation:

$a_1 + 2a_2 + 2^2 a_3 + \ldots + 2^{n-1} a_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$ for each $n \geq 1$.

Based on the $a_{n+1} - 2 = \dfrac{-(a_n - 2)^2}{a_n^2 - 2a_n + 2} \leq 0$ relation each term of the sequence is less or equal than 2 starting from the 3rd term. The equality stands for the a=4 initial value. Furthermore since $a_{n+1} - a_n = \dfrac{-a_n(a_n - 1)(a_n - 2)}{a_n^2 - 2a_n + 2}$, if $a_2 > 1$, $a > 2$, then the sequence has all terms numbers between 1 and 2 starting from the 3rd term, severely incresing, convergent, the limit value being 2.

For $a < 2$, then a_2 can be negativ, or a value between 0 and 1 , but the a_3 value surely is less than 1, being a positive number and similarly the following terms of the sequence. In this case the sequence is severely decreasing, convergent and the limit value is 0 .

We can determinate the general term of the sequence. The recursive relation can be written as $\dfrac{1}{a_{n+1}} = 1 - \dfrac{2}{a_n} + \dfrac{2}{a_n^2}$ and multiplied by 2 and transforming it we get: $\dfrac{2}{a_{n+1}} - 1 = \left(\dfrac{2}{a_n} - 1\right)^2$,

from that results $\dfrac{2}{a_n} - 1 = \left(\dfrac{2}{a_2} - 1\right)^{2^{n-2}}$, for each $n \geq 3$.

Replacing the $a_2 = \dfrac{a}{a-2}$ value ,we found: $a_n = \dfrac{2a^{2^{n-2}}}{a^{2^{n-2}} + (a-4)^{2^{n-2}}}$, for each $n \geq 2$. The examination of this sequence was setting as task on the XVII Transilvanien Hungarien Mathematics Competition in Miercurea Ciuc, 2007, suggested by Kovács Béla, presented in : Matlap 2007/5, page: 32. [9].

2.3. Thirdly let us take the p=3 case. It seems to be a more complicated and mysterious case.

Now the sequence is : $a_1 = a \neq 3$, $a_2 = \dfrac{a}{a-3}$, $a_{n+1} = \dfrac{a_n^2}{a_n^2 - 3a_n + 3}$, fo each $n \geq 2$. Always defined, having all the terms positive numbers from the 3rd term. Is valid the next relation : $a_1 + 3a_2 + 3^2 a_3 + ... + 3^{n-1} a_n = a_1 \cdot a_2 \cdot a_3 \cdot ... \cdot a_n$ for each $n \geq 1$. In this case , since $a_{n+1} - 4 = \dfrac{-3(a_n-2)^2}{a_n^2 - 3a_n + 3} \leq 0$ all members of the sequence is less or equal than 4 , for each $n > 3$. In addition $a_{n+1} - 2 = \dfrac{3 - (a_n-3)^2}{a_n^2 - 3a_n + 3}$ and if $2 < a_n < 4$, then $2 < a_{n+1} < 4$, (mathematical induction). If $a > 3$, then $a_2 > 1$, and each term of the sequence is greater than 1. Since $a_{n+1} - a_n = \dfrac{-a_n(a_n-1)(a_n-3)}{a_n^2 - 3a_n + 3}$, $n \geq 2$, we can remark, that the sequence is severely increasing by the time one of its members reach the 3 value, or a greater value. Based on the $a_{n+1} - 3 = \dfrac{-(a_n-3)(2a_n-3)}{a_n^2 - 3a_n + 3}$ relation, if $a_n > 3$, then $a_{n+1} < 3$, and vice versa (starting from any term).

For the $a = \dfrac{9}{2}$ and $a = 9$ initial values each term of the sequence is equal with 3 staring from the 2.nd respective 3.rd member. Using another initial values we found that each member of the sequence is equal with 3 starting from a certain member.

The general definition of this values is an open-question till now. Finally it results from the

$$a_{n+2} - a_n = \frac{-a_n(a_n-1)(a_n-3)^3}{a_n^4 - 9(a_n-1)(a_n^2 - 3a_n + 3)} \ , \ n \geq 2 \ \text{context} - \text{since the denominator is always}$$

positive- that the the members of the sequence with an even rank are growing strictly and the members of a sequence with an odd rank decrease strictly, or reverse (starting from one of the members). Both part sequence are convergent, their common limit value being 3., which is the limit value of the sequence too. If we have $a < 3$, then the terms of the sequence are numbers between 0 and 1 beginning from the 3rd member, in this case the sequence decrease strictly, is convergent, and the limit value is 0.

The definition of the general member of the sequence is an open question.

Comment: Introducing the $g(a) = \dfrac{a}{a-3}$ and $f(a) = \dfrac{a^2}{a^2 - 3a + 3}$ functions, the sequence can be written as follows: $a_1 = a \neq 3$, $a_2 = g(a)$ and

$a_n = \underbrace{f \circ f \circ \ldots \circ f}_{n-2}(g(a))$, $n \geq 3$, but this cannot be considered as the general member's

closed form.

2.4 In the fourth case let us take the p = 4. This already is a very complicated, special and mysterious case.

Now the sequence is: $a_1 = a \neq 4$, $a_2 = \dfrac{a}{a-4}$, $a_{n+1} = \dfrac{a_n^2}{a_n^2 - 4a_n + 4}$, for each $n \geq 2$, not

always defined, but having all his terms strictly positive beginning with the 3rd term.

The next relation is valid in the domain of interpretation of the sequence $a_1 + 4a_2 + 4^2 a_3 + \ldots + 4^{n-1} a_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$ for each $n \geq 1$. If a = 8 , then the 2nd term of the sequence is 2 and the 3rd is not defined. For a = $8(2 \pm \sqrt{2})$, we have the 3rd term being 2,

but the 4th term alredy is not defined. Based on the recursive relation: $a_{n+1} = \dfrac{a_n^2}{(a_n - 2)^2}$, if

one member of the sequence has the value 2, then the next members are already not defined. This exists on the case of very much initial value. We deal with this later.

For $a < 4$ the a_2 can be negative, or a number between the 0 and 1 values, but the a_3 is already certainly less then 1, being a positive number, and in the same way the additional

members of the sequence. Since $a_{n+1} - a_n = \dfrac{-a_n(a_n-1)(a_n-4)}{a_n^2 - 4a_n + 4} < 0$ the sequence

decrease strictly, is convergent, and the limit value is 0.

When a > 4 then we define he general term of the sequence using a bold, special undertaking. In this case $a_2 > 1$ and so are the other terms of the sequence. Transforming the recursive relation we get: $\dfrac{1}{a_{n+1}} = 1 - \dfrac{4}{a_n} + \dfrac{4}{a_n^2} \Leftrightarrow 1 - \dfrac{2}{a_{n+1}} = 1 - 2\left(1 - \dfrac{2}{a_n}\right)^2$. For n = 2

results $1 - \dfrac{2}{a_3} = 1 - 2\left(1 - \dfrac{2}{a_2}\right)^2$. Since $a_2 = \dfrac{a}{a-4} > 1$, we have that $-1 < 1 - \dfrac{2}{a_2} < 1$, so it

exist an $\alpha \in [0, \pi)$, so that $\cos\alpha = 1 - \dfrac{2}{a_2}$, that is $\cos\alpha = \dfrac{8-a}{a} \Leftrightarrow \alpha = \arccos\dfrac{8-a}{a}$. In

this case, based on the well known $2\cos^2\alpha - 1 = \cos 2\alpha$ relation we get $a_3 = \dfrac{2}{1+\cos 2\alpha}$

and and we receive applying of the mathematical induction, that $a_n = \dfrac{2}{1+\cos 2^{n-2}\alpha}$ for

each $n \geq 3$. So the general term of the sequence is $a_n = \dfrac{2}{1+\cos(2^{n-2}\arccos\dfrac{8-a}{a})}$, which

can be witten as $a_n = \dfrac{1}{\cos^2(2^{n-3}\arccos\dfrac{8-a}{a})}$, or as $a_n = 1 + \mathrm{tg}^2(2^{n-3}\arccos\dfrac{8-a}{a})$ being

valid for each $n \geq 3$. From this we may establish those initial values already, for which the sequence is not defined. These will be definied by the solution of the $1 + \cos(2^{n-2}\arccos\dfrac{8-a}{a}) = 0$ equation. Results: $\arccos\dfrac{8-a}{a} = \dfrac{(2k+1)\pi}{2^{n-2}} \Rightarrow$

$\dfrac{8-a}{a} = \cos\dfrac{(2k+1)\pi}{2^{n-2}} \Rightarrow a = \dfrac{8}{1+\cos\dfrac{(2k+1)\pi}{2^{n-2}}}$, where $k \in Z$ és $0 \leq k < 2^{n-2}$.

Depending by the initial value, the sequence may be a constant, from a certain member starting or may be periodic, but also it may be terms mysteriounly random in sequences with an optional member, what is not convergent naturally.

For instance: for $a = \dfrac{16}{3}$ the sequence $\dfrac{16}{3}$, 4, 4, 4, 4, 4, . . . , is constant

For a = 16 the sequence: 16, $\dfrac{4}{3}$, 4, 4, 4, 4, ,. . . , is constant,

For $a = 16(2 - \sqrt{3})$ the sequence looks like: $16(2 - \sqrt{3})$, $4(2 + \sqrt{3})$, $\dfrac{4}{3}$, 4, 4, 4, . . .

In case of $a_1 = a = \dfrac{8}{1+\cos\dfrac{\pi}{5}} = \dfrac{32}{5+\sqrt{5}}$ the terms of the sequence are $2(3\pm\sqrt{5})$, alternate,

periodic sequence. We can receive other periodic sequence on the case of other initial values

For a = 5 we have: $a_1 = a_2 = 5$ and $a_n = 1 + \text{tg}^2(2^{n-3}\arccos\dfrac{3}{5})$, for each $n \geq 3$, a special

and mysterious sequence.

For a = 6 the sequence: $6, 3, 9, \dfrac{81}{49}, \dfrac{6561}{289}, \dfrac{43046721}{35796289}, \ldots$, with approaching values :

6 , 3 , 9 , 1.653 , 22.7 , 1.2 , 2.274 , 68.869 , 1.06 , 1.275 , . . . etc. The general member is: a_n

$= 1 + \text{tg}^2(2^{n-3}\arccos\dfrac{1}{3})$, for each $n \geq 3$.

The sequence can be given in trigonometrical form with the $a_1 = \dfrac{8}{1+\cos\alpha}$ initial value as

follows: $\dfrac{8}{1+\cos\alpha}$, $\dfrac{2}{1-\cos\alpha}$, $\dfrac{1}{\cos 2\alpha}$, $\dfrac{1}{\cos 4\alpha}$, $\dfrac{1}{\cos 8\alpha}$, . . . etc. Tha valid relation is:

$$\dfrac{16}{\sin^2\alpha} + \sum_{k=3}^{n}\dfrac{4^{k-1}}{\cos^2 2^{k-3}\alpha} = \dfrac{16}{\sin^2\alpha}\cdot\prod_{k=3}^{n}\dfrac{1}{\cos^2 2^{k-3}\alpha} = \dfrac{4^n}{\sin^2 2^{n-2}\alpha}$$, for each $n \geq 3$, in the

domain of interpretation of the sequence. (for $\alpha \neq \dfrac{(2k+1)\pi}{2^{n-3}}$)

The definition of the general member of the sequence for a < 4 remain an open question.

3. Comments and open, undetermined questions

It is remarkable that depending on the p parameter and on the initial value on the case of his different values what kind of special and interesting, maybe mysterious sequence we get. It would be interesting to examinate some similar cases. We may get interesting sequence in case of p being a negative value. When our sequence is convergent, constant, or periodic sequence? Is definable the general member of the sequence in other specific cases? How the members of the sequence are scattered when he is not periodic and not convergent, or his limit values not concerned? On the case of what kind of initial values are not defined the sequence? To answer this questions the additional examination of the sequence is necessary. I am sure, that very many opportunities not exploited offer themselves yet. I draw the attention of interested ones to it for the topic.

4. Finally, a few other similar cases, the prof of which I leave to interested readers.

4.1. Now the sequence has the following form: $a_1 = a \neq 2$, $a_2 = \dfrac{a}{a-2}$ and

$$a_{n+1} = \frac{na_n^2}{na_n^2 - (n+1)a_n + n + 1} \quad , \text{ for each } n \geq 2.$$

Is valid the next relation: $a_1 + 2a_2 + 3a_3 + \ldots + na_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \quad \forall \, n \geq 1$.

4.2. Now the sequence has the following form: $a_1 = a \neq \dfrac{1}{2}$, $a_2 = \dfrac{2a}{2a-1}$ and

$$a_{n+1} = \frac{(n+1)a_n^2}{(n+1)a_n^2 - na_n + n} \quad , \text{ for each } n \geq 2$$

Is valid the next relation: $a_1 + \dfrac{1}{2}a_2 + \dfrac{1}{3}a_3 + \ldots + \dfrac{1}{n}a_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \qquad \forall \, n \geq 1$.

4.3. Now the sequence has the following form: $a_1 = a \in Q - \{2\}$, $a_2 = \dfrac{a}{a-2}$ and

$$a_{n+1} = \frac{a_n^2}{a_n^2 - (n+1)a_n + n + 1} \quad , \text{ for each } n \geq 2.$$

Is valid the next relation: $a_1 + 2!\, a_2 + 3!\, a_3 + \ldots + n!\, a_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \quad \forall \, n \geq 1$.

4.4. Now the sequence has the following form: $a_1 = a \neq \dfrac{1}{2}$, $a_2 = \dfrac{2a}{2a-1}$ and

$$a_{n+1} = \frac{(n+1)a_n^2}{(n+1)a_n^2 - a_n + 1} \quad , \text{ for each } n \geq 2$$

Is valid the next relation: $a_1 + \dfrac{1}{2!}a_2 + \dfrac{1}{3!}a_3 + \ldots + \dfrac{1}{n!}a_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \qquad \forall \, n \geq 1$.

4.5. Now the sequence has the following form: $a_1 = a \neq \dfrac{1}{2}$, $a_2 = \dfrac{a}{2a-1}$ and

$$a_{n+1} = \frac{na_n^2}{(n+1)a_n^2 - na_n + n - 1} \quad , \text{ for each } n \geq 2$$

Is valid the next relation: $a_1 + a_2 + a_3 + \ldots + a_n = n \cdot a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \quad \forall \, n \geq 1$.

4.6. Now the sequence has the following form: $a_1 = a \neq 2$, $a_2 = \dfrac{2a}{a-2}$ and

$$a_{n+1} = \frac{(n^2-1)a_n^2}{n(n-1)a_n^2 - (n^2-1)a_n + n(n+1)} \quad , \text{ for each } n \geq 2$$

Is valid the next relation: $n \cdot (a_1 + a_2 + a_3 + \ldots + a_n) = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \qquad \forall \, n \geq 1$.

4.7. Now the sequence has the following form: $a_1 = a \neq 2$, $a_2 = \dfrac{2a}{a-2}$ and

$$a_{n+1} = \frac{2a_n^2}{a_n^2 - 2a_n + 4} \ , \quad \text{for each } n \geq 2$$

Is valid the next relation: $2^{n-1} \cdot (a_1 + a_2 + a_3 + \ldots + a_n) = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \quad \forall\, n \geq 1$.

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