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TEN APPLICATIONS FOR IONESCU- WEITZENBÖCK'S INEQUALITY

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Abstract. This paper presents refinements and new solutions for some problems - published by math journals from all over the world related to Ionescu-Weitzenböck inequality .

INTRODUCTION

Ion Ionescu discovered with 22 years before Weitzenböck the inequality $a^2+b^2+c^2\geq 4\sqrt{3}S$

The author of this paper along with prof. D. M. Bătineţu-Giurgiu demonstrated in Romanian Mathematical Gazette , No. 1/2013, pp. 1-10, that the Weitzenböck's inequality must be named the Ionescu-Weitzenböck's inequality.

Our proof is based on: Romanian Mathematical Gazette, Vol. III (15 September 1897 – 15 August 1898), No. 2 , 15 October 1897, on page 52, Ion Ionescu, the founder of Romanian Mathematical Gazette, published problem 273: prove that there is no triangle for which the inequality $4S\sqrt{3}>a^2+b^2+c^2$ **can be satisfied. The solution of the problem 273, apperead in Romanian Mathematical Gazette, Vol. III (15 September 1897 – 15 August 1898), No. 12 , 15 August 1898, on pages 281, 282 and 283. In the year 1919, Roland Weitzenböck publishead in Mathematische Zeitschrift, Vol. 5, No. 1-2, pp. 137-146 the article Uber eine Ungleichung in der Dreiecksgeometrie, where he proof that: In any triangle** ABC **, with usual notations holds the inequality:** $a^2+b^2+c^2\geq 4\sqrt{3}S$ **.**

MAIN RESULTS

Application 1 (D.M. Bătineţu-Giurgiu , N. Stanciu - La Gaceta de la RSME, No. 2/2020, problem 397).
Prove that in any triangle ABC **with the area** F **, the medians** m_a,m_b,m_c **(** m_a **is the mediane from the vertex** A **,** m_b **is the mediane from the vertex** B **,** m_c **is the mediane from the vertex** C **) and usual notations is true the following inequality**

$$\frac{\sqrt{3}}{2}(a^2+b^2+c^2)\geq am_a+bm_b+cm_c\geq 6F\ .$$

Solution. First we prove that $a^2+b^2+c^2\geq 2\sqrt{3}\cdot a\cdot m_a$, (1). Indeed,

$$a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a \Leftrightarrow (a^2+b^2+c^2)^2 \ge 12\cdot a^2\cdot m_a^2 = 3a^2\cdot 4m_a^2 = 3a^2(2b^2+2c^2-a^2)$$
$$\Leftrightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 \ge 6a^2b^2+6c^2a^2-3a^4$$
$$\Leftrightarrow 4a^4+b^2+c^2-4a^2b^2-4c^2a^2+2b^2c^2 \ge 0$$
$\Leftrightarrow (2a^2-b^2-c^2)^2 = (b^2+c^2-2a^2)^2 \ge 0$, true. Therefore, $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a$; $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot b\cdot m_b$ and $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot c\cdot m_c$ which by adding up yielding that $3(a^2+b^2+c^2) \ge 2\sqrt{3}(am_a+bm_b+cm_c)$, (2). Next we have $2\sqrt{3}(am_a+bm_b+cm_c) \ge 2\sqrt{3}(ah_a+bh_b+ch_c) = 12\sqrt{3}F$, (3), where we denote h_a the altitude from the vertex A, h_b the altitude from the vertex B, h_c the altitude from the vertex C. From (2) and (3) we obtain the desired inequality.

Application 2. (D.M. Bătineţu-Giurgiu , N. Stanciu- Refinement of Ionescu-Weitzenböck inequality).

Prove that in any triangle ABC with the area F, the medians m_a, m_b, m_c (m_a is the mediane from the vertex A, m_b is the mediane from the vertex B, m_c is the mediane from the vertex C and usual notations is true the following inequality

$$a^2+b^2+c^2 \ge \frac{2}{\sqrt{3}}(am_a+bm_b+cm_c) \ge 4\sqrt{3}F.$$

Solution. First we prove that $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a$, (1). Indeed,

$$a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a \Leftrightarrow (a^2+b^2+c^2)^2 \ge 12\cdot a^2\cdot m_a^2 = 3a^2\cdot 4m_a^2 = 3a^2(2b^2+2c^2-a^2)$$
$$\Leftrightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 \ge 6a^2b^2+6c^2a^2-3a^4$$
$$\Leftrightarrow 4a^4+b^2+c^2-4a^2b^2-4c^2a^2+2b^2c^2 \ge 0$$
$\Leftrightarrow (2a^2-b^2-c^2)^2 = (b^2+c^2-2a^2)^2 \ge 0$, true.

Therefore, $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a$; $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot b\cdot m_b$; $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot c\cdot m_c$ which by adding up yielding that $3(a^2+b^2+c^2) \ge 2\sqrt{3}(am_a+bm_b+cm_c)$, (2).

Next we have $2\sqrt{3}(am_a+bm_b+cm_c) \ge 2\sqrt{3}(ah_a+bh_b+ch_c) = 12\sqrt{3}F$, (3),

where we denote h_a the altitude from the vertex A, h_b the altitude from the vertex B, h_c the altitude from the vertex C. From (2) and (3) we obtain the desired inequality.

Application 3. (D.M. Bătineţu-Giurgiu , N. Stanciu - Other refinement of Ionescu-Weitzenböck inequality).

Prove that in any triangle ABC with the area F, the altidudes h_a, h_b, h_c, the interior angles bisectors w_a, w_b, w_c and usual notations is true the following inequality

$$a^2+b^2+c^2 \ge \frac{4F}{\sqrt{3}}\left(\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\right) \ge 4\sqrt{3}F.$$

Solution. WLOG we can assume that $a \le b \le c$, then $w_a \ge w_b \ge w_c$ and $h_a \ge h_b \ge h_c$.

By Chebyshev's inequality we get that $\sum_{cyclic}\frac{w_a}{h_a} \le \frac{1}{3}\left(\sum_{cyclic} w_a\right)\left(\sum_{cyclic}\frac{1}{h_a}\right)$, (1);

$\sum_{cyclic}\frac{1}{h_a} = \sum_{cyclic}\frac{a}{ah_a} = \frac{1}{2F}(a+b+c) = \frac{s}{F}$, (2), and,

$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2bc}{b+c}\sqrt{\frac{s(s-a)}{bc}} = \frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)} \le \frac{2\sqrt{bc}}{2\sqrt{bc}}\sqrt{s(s-a)} = \sqrt{s(s-a)}\ , (3)$$

and other two similar. Using the inequality $x^2+y^2+z^2 \ge \frac{(x+y+z)^2}{3}$, (*), and (3) we deduce that

$$\sum_{cyclic} s(s-a) \ge \frac{\left(\sum_{cyclic}\sqrt{s(s-a)}\right)^2}{3} \Leftrightarrow 3s^2 \ge \left(\sum_{cyclic}\sqrt{s(s-a)}\right)^2 \Leftrightarrow \sum_{cyclic}\sqrt{s(s-a)} \le s\sqrt{3} \Leftrightarrow$$

$\Leftrightarrow \sum_{cyclic} w_a \le s\sqrt{3}$, (4). From (1), (2), (4) and (*) we obtain that

$$\sum_{cyclic}\frac{w_a}{h_a} \le \frac{1}{3}\left(\sum_{cyclic} w_a\right)\left(\sum_{cyclic}\frac{1}{h_a}\right) \le \frac{1}{3}\cdot s\sqrt{3}\cdot\frac{s}{F} = \frac{(a+b+c)^2}{4F\sqrt{3}} \le \frac{3(a^2+b^2+c^2)}{4F\sqrt{3}}, (5).$$

Since $w_a \ge h_a, w_b \ge h_b, w_c \ge h_c$ from (5) we get

$$3 \le \sum_{cyclic}\frac{w_a}{h_a} \le \frac{3(a^2+b^2+c^2)}{4F\sqrt{3}} \Leftrightarrow a^2+b^2+c^2 \ge \frac{4F}{\sqrt{3}}\left(\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\right) \ge 4\sqrt{3}F, \text{ Q.E.D.}$$

Application 4. (D.M. Bătineţu-Giurgiu , N. Stanciu - Math Problems, Volume 3, Issue 1, Junior MathProblems, Problem 1, 2013).

Prove that in all triangle ABC, with usual notations, holds:

$$\frac{a^3}{b\cdot R+c\cdot r}+\frac{b^3}{c\cdot R+a\cdot r}+\frac{c^3}{a\cdot R+b\cdot r} \ge \frac{4\sqrt{3}}{R+r}S.$$

Solution. We have $V = \sum_{cyc}\frac{a^3}{b\cdot R+c\cdot r} = \sum_{cyc}\frac{a^4}{abR+acr} \ge 2\cdot\sum_{cyc}\frac{(a^2)^2}{(a^2+b^2)R+(a^2+c^2)r}$,

where we apply *Bergström* 's inequality and we deduce that:

$$V \geq 2 \cdot \frac{\left(\sum_{cyc} a^2\right)^2}{\sum_{cyc} R(a^2+b^2)+\sum_{cyc} r(a^2+c^2)} = 2 \cdot \frac{\left(\sum_{cyc} a^2\right)^2}{2(R+r)\cdot \sum_{cyc} a^2} = \frac{\sum_{cyc} a^2}{R+r}.$$

Then applying Ionescu- Weitzenböck' s inequality, i.e $\sum_{ciclic} a^2 \geq 4S\sqrt{3}$, $V \geq \frac{4S\sqrt{3}}{R+r}$, and we are done.

Application 5. (D.M. Bătineţu-Giurgiu , N. Stanciu - Revista Escolar de la Olimpiada Iberoamericana de Matematica, Numero 49 (julio-agosto 2013), Problema 242).

If $m \in R_+$, then in all triangle ABC, with usual notations (i.e. R - circumradius, r - inradius, the lengths of the sides are a,b,c and S - the area of triangle ABC) the following inequality holds:

$$\frac{a^{m+2}}{(b\cdot R+c\cdot r)^m}+\frac{b^{m+2}}{(c\cdot R+a\cdot r)^m}+\frac{c^{m+2}}{(a\cdot R+b\cdot r)^m} \geq \frac{4\sqrt{3}}{(R+r)^m}S.$$

Solution. $W = \sum_{cyc} \frac{a^{m+2}}{(b\cdot R+c\cdot r)^m} = \sum_{cyc} \frac{a^{2(m+1)}}{(abR+acr)^m} \geq 2^m \cdot \sum_{cyc} \frac{(a^2)^{m+1}}{((a^2+b^2)R+(a^2+c^2)r)^m}$, and applying the inequality of *J. Radon* we obtain

$$W \geq 2^m \cdot \frac{\left(\sum_{ciclic} a^2\right)^{m+1}}{(\sum_{ciclic} R(a^2+b^2)+\sum_{ciclic} r(a^2+c^2))^m} = 2^m \cdot \frac{\left(\sum_{ciclic} a^2\right)^{m+1}}{2^m(R+r)^m \cdot (\sum_{ciclic} a^2)^m} = \frac{\sum_{ciclic} a^2}{(R+r)^m}.$$

By *Ionescu- Weitzenböck*'s inequality, i.e. $\sum_{cyc} a^2 \geq 4S\sqrt{3}$, we get $W \geq \frac{4S\sqrt{3}}{(R+r)^m}$, and the proof is complete.

Application 6. (D.M. Bătineţu-Giurgiu , N. Stanciu - Math Problems, Volume 3, Issue 1, Junior MathProblems, Problem 14, 2013) . If $m \in [0,\infty)$, $x,y,z,t \in (0,\infty)$, then in any triangle ABC, with usual notations holds $\sum_{cyclic} \frac{\left(xm_a^2+ym_b^2\right)^{m+1}}{\left(zw_a^2+tw_b^2\right)^m} \geq 3\cdot\sqrt{3}\cdot\frac{(x+y)^{m+1}}{(z+t)^m}S$.

Solution. $w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \Rightarrow w_a \leq \sqrt{s(s-a)} \Rightarrow w_a^2 \leq s(s-a)$, and other two similar.

So, $w_a^2+w_b^2+w_c^2 \le s(s-a+s-b+s-c)=s(3s-2s)=s^2=$

$=\frac{(a+b+c)^2}{4}\le \frac{3}{4}(a^2+b^2+c^2)=m_a^2+m_b^2+m_c^2$, (1); by $\sum_{cyclic} m_a^2=\frac{3}{4}\sum_{cyclic} a^2$, J. Radon's inequality and (1), we obtain $\sum_{cyclic}\frac{(xm_a^2+ym_b^2)^{m+1}}{(zw_a^2+tw_b^2)^m}\overset{RADON}{\ge}\frac{\left(\sum_{cyclic}(xm_a^2+ym_b^2)\right)^{m+1}}{\left(\sum_{cycclic}(zw_a^2+tw_b^2)\right)^m}=$

$$=\frac{\left((x+y)\sum_{cyclic} m_a^2\right)^{m+1}}{\left((z+t)\sum_{cycclic} w_a^2\right)^m}\ge\frac{(x+y)^{m+1}\left(\sum_{cyclic} m_a^2\right)^{m+1}}{(z+t)^m\left(\sum_{cyclic} m_a^2\right)^m}=\frac{(x+y)^{m+1}\left(\sum_{cyclic} m_a^2\right)}{(z+t)^m}=\frac{3}{4}\cdot\frac{(x+y)^{m+1}}{(z+t)^m}\sum_{cyclic} a^2 \text{ , (2),}$$

by Ion Ionescu – Weitzenböck inequality we have $a^2+b^2+c^2\ge 4\sqrt{3}S$, (3). By (2) and (3) we obtain

$$\sum_{cyclic}\frac{(xm_a^2+ym_b^2)^{m+1}}{(zw_a^2+tw_b^2)^m}\ge 3\cdot\sqrt{3}\cdot\frac{(x+y)^{m+1}}{(z+t)^m}S \text{ , q.e.d.}$$

Application 7. (D.M. Bătineţu-Giurgiu , N. Stanciu - Recreaţii Matematice, 2/2013).

Prove that in all triangle ABC, with usual notations, holds

$$\frac{m_a^3}{R\cdot m_b+r\cdot m_c}+\frac{m_b^3}{R\cdot m_c+r\cdot m_a}+\frac{m_c^3}{R\cdot m_a+r\cdot m_b}\ge\frac{3\sqrt{3}}{R+r}S.$$

Solution. $U=\sum_{cyc}\frac{m_a^3}{R\cdot m_b+r\cdot m_c}=\sum_{cyc}\frac{(m_a^2)^2}{R\cdot m_a\cdot m_b+r\cdot m_a\cdot m_c}\ge$

$\ge 2\cdot\sum_{cyc}\frac{(m_a^2)^2}{R(m_a^2+m_b^2)+r(m_a^2+m_c^2)}$, where we apply *Bergström's* inequality and well-known formula $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$. We obtain that

$$U\ge 2\cdot\frac{\left(\sum_{cyc} m_a^2\right)^2}{R\cdot\sum_{cyc}(m_a^2+m_b^2)+r\cdot\sum_{cyc}(m_a^2+m_c^2)}=\frac{2\cdot\left(\sum_{cyc} m_a^2\right)^2}{2R\cdot\sum_{cyc} m_a^2+2r\cdot\sum_{cyc} m_a^2}=$$

$= \frac{\sum_{cyc} m_a^2}{R+r} = \frac{3}{4} \cdot \frac{a^2+b^2+c^2}{R+r}$, where we use the *Ionescu – Weitzenböck* inequality, i.e. $a^2+b^2+c^2 \geq 4S\sqrt{3}$, and we deduce that $U \geq \frac{3}{4} \cdot \frac{1}{R+r} \cdot 4S\sqrt{3} = \frac{3\sqrt{3}}{R+r} S$, and we are done.

Application 8. (D.M. Bătineţu-Giurgiu , N. Stanciu - The College Mathematics Journal (CMJ), Vol. 45, No. 5, November 2014, Problem 1038).

If $m \in R_+, x, y \in R_+^*$**, then in any triangle** ABC **holds**

$$\frac{a^{m+2}}{(xb+yc)^m} + \frac{b^{m+2}}{(xc+ya)^m} + \frac{c^{m+2}}{(xa+yb)^m} \geq \frac{4\sqrt{3}}{(x+y)^m} \cdot areaABC \text{ .}$$

Solution. $E = \sum_{cyclic} \frac{a^{m+2}}{(xb+yc)^m} = \sum_{cyclic} \frac{a^{2(m+1)}}{(xab+yac)^m} \overset{(RADON)}{\geq} \frac{(a^2+b^2+c^2)^{m+1}}{\left(\sum_{cyclic}(xab+yac)\right)^m} =$

$= \frac{(a^2+b^2+c^2)^{m+1}}{(x+y)^m(ab+bc+ca)^m}$, and by $a^2+b^2+c^2 \geq ab+bc+ca$, we deduce

$$E \geq \frac{(a^2+b^2+c^2)^{m+1}}{(x+y)^m(a^2+b^2+c^2)^m} = \frac{a^2+b^2+c^2}{(x+y)^m}, \text{ (1).}$$

Ionescu-Weitzenböck inequality $a^2+b^2+c^2 \geq 4\sqrt{3} \cdot areaABC$, (2)

From (1) and (2) we obtain the desired inequality.

Application 9. (D.M. Bătineţu-Giurgiu , N. Stanciu - The College Mathematics Journal (CMJ), Vol. 46, No. 2, March 2015, Problem 1050).

Let $A_1A_2...A_n, n \geq 3,$ **be a convex polygon with** a_k **the length of the side** $[A_kA_{k+1}], k = \overline{1,n},$ $A_{n+1} = A_1$**, and let** S **be the area of the polygon. Prove that**

$$\left(\sum_{k=1}^n a_k^{2m+4}\right)\left(\sum_{k=1}^n \frac{1}{a_k^{2m}}\right) \geq 16S^2 tg^2 \frac{\pi}{n}, \ \forall m \in R_+ .$$

Solution. By *J. Radon's* inequality we deduce that: $\sum_{k=1}^n a_k^{2m+4} \geq \frac{1}{n^{m+1}}\left(\sum_{k=1}^n a_k^2\right)^{m+2}, \forall m \in R_+$, (1),

and also by *J. Radon's* inequality we obtain that $\sum_{k=1}^n \frac{1}{a_k^{2m}} \geq \frac{n^{m+1}}{\left(\sum_{k=1}^n a_k^2\right)^m}, \forall m \in R_+$, (2).

So, (1) and (2) yields that $U_n = \left(\sum_{k=1}^{n} a_k^{2m+4}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k^{2m}}\right) \ge \left(\sum_{k=1}^{n} a_k^2\right)^2$, $\forall m \in R_+$, (3).

By *E. Just and N. Schaumberger'* s inequality (The Problem 1634 from AMM, 70(1963)) we have that $\sum_{k=1}^{n} a_k^2 \ge 4S \cdot tg\frac{\pi}{n}$, (4). Therefore, from (3) and (4) we obtain that $U_n \ge 16S^2 tg^2 \frac{\pi}{n}$, $\forall m \in R_+$ and we are done.

Application 10. (D.M. Bătineţu-Giurgiu , N. Stanciu - Revista Escolar de la Olimpiada Iberoamericana de Matematica, Problem 277, August 2016) . Show that in any triangle ABC (with usual notations) holds the following inequality

$$(ab+bc+ca)^2 + 2(a^2+b^2+c^2)^2 \ge 16\sqrt{3} \cdot s^3 \cdot r \text{.}$$

Solution. For any $w_1, w_2, w_3 \in R$ we have $w_1^2 + w_2^2 + w_3^2 \ge w_1 \cdot w_2 + w_2 \cdot w_3 + w_3 \cdot w_1$, (1), with equality iff $w_1 = w_2 = w_3$. If $m, n, p \in R_+^*$, then $\frac{m \cdot n}{p} + \frac{n \cdot p}{m} + \frac{p \cdot m}{n} \ge m+n+p$, (2) with equality iff $m=n=p$. Indeed if in (1) we take $w_1 = \sqrt{\frac{m \cdot n}{p}}$, $w_2 = \sqrt{\frac{n \cdot p}{m}}$, $w_3 = \sqrt{\frac{p \cdot m}{n}}$ yields (2).

If $t, u, v, x, y, z \in R_+^*$, then $\frac{(tx+uy+vz)(ty+uz+vx)}{tz+ux+vy} + \frac{(ty+uz+vx)(tz+ux+vy)}{tx+uy+vz} +$
$+\frac{(tz+ux+vy)(tx+uy+vz)}{ty+uz+vx} \ge (t+u+v)(x+y+z)$, (3). Indeed if in (2) we take $m = tx+uy+vz$, $n = ty+uz+vx$, $p = tz+ux+vy$ yields (3).

Lemma. For any $x, y, z \in R_+^*$, holds

$$(xy+yz+zx)^2 + 2(x^2+y^2+z^2)^2 \ge (x^2+y^2+z^2)(x+y+z)^2, \text{ (4).}$$

Proof. In (3) we take $t=x, u=y, v=z$ and we obtain

$$\frac{(x^2+y^2+z^2)(xy+yz+zx)}{xz+xy+zy} + \frac{(xy+yz+zx)(xz+xy+yz)}{x^2+y^2+z^2} +$$
$$+\frac{(xz+xy+yz)(x^2+y^2+z^2)}{xy+yz+xz} \ge (x+y+z)^2 \Leftrightarrow \frac{(xy+yz+zx)^2}{x^2+y^2+z^2} + 2(x^2+y^2+z^2) \ge (x+y+z)^2$$

$\Leftrightarrow (xy+yz+zx)^2 + 2(x^2+y^2+z^2)^2 \ge (x+y+z)^2(x^2+y^2+z^2)$, q.e.d.

If in (4) we take $x=a, y=b, z=c$ we obtain

$$(ab+bc+ca)^2 + 2(a^2+b^2+c^2)^2 \ge (a+b+c)^2(a^2+b^2+c^2), \text{ (5).}$$

By *Ionescu-Weitzenböck* 's inequality we have $a^2+b^2+c^2 \ge 4\sqrt{3} \cdot S$, (I-W).

So (5) becomes $(ab+bc+ca)^2 + 2(a^2+b^2+c^2)^2 \ge 4s^2 \cdot 4\sqrt{3}S = 16\sqrt{3} \cdot s^2 \cdot sr = 16\sqrt{3}s^3 r$.