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3701. In ΔABC the following relationship holds:

$$\sum \frac{h_a}{(s-a)^2-\left(n_a-\sqrt{4r^2+(b-c)^2}\right)^2}=\frac{4R+r}{2r^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Tapas Das-India

$$\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}, n_a^2=s^2-2r_ah_a$$

$$(Refference: Bogdan\ Fustei$$
$$-ABOUT\ NAGEL\ AND\ GERGONNE\ CEVIANS, www.ssmrmh.ro)$$

$$Known\ results: \sum \frac{1}{s-a}=\frac{4R+r}{sr}, r_a=\frac{F}{s-a}$$

$$n_a-\sqrt{4r^2+(b-c)^2}=n_a-\frac{n_a}{h_a}.2r=n_a\left(1-\frac{a}{s}\right)=\frac{n_a(s-a)}{s}$$

$$(s-a)^2-\left(n_a-\sqrt{4r^2+(b-c)^2}\right)^2=(s-a)^2-\frac{n_a^2(s-a)^2}{s^2}=$$

$$=\frac{(s-a)^2}{s^2}(s^2-n_a^2)=\frac{(s-a)^2}{s^2}2r_ah_a$$

$$\frac{h_a}{(s-a)^2-\left(n_a-\sqrt{4r^2+(b-c)^2}\right)^2}=\frac{h_a}{\frac{(s-a)^2}{s^2}2r_ah_a}=\frac{s^2}{2}\frac{1}{(s-a)^2r_a}=\frac{s^2}{2F}\frac{.1}{s-a}$$

$$\sum \frac{h_a}{(s-a)^2-\left(n_a-\sqrt{4r^2+(b-c)^2}\right)^2}=\frac{s^2}{2F}\sum \frac{1}{s-a}=\frac{s}{2r}\frac{4R+r}{sr}=\frac{4R+r}{2r^2}$$

3702. If $\mathrm{I}\to$ incenter in $any\ \Delta$ ABC and $\mathrm{I}(x,\mathrm{y})=\frac{1}{\mathrm{e}}\left(\frac{\mathrm{y}^{\mathrm{y}}}{x^x}\right)^{\frac{1}{\mathrm{y}-x}}, 0<x<$ y, then :

$$\sum_{\mathrm{cyc}} \mathrm{I}\left(\sqrt{\frac{\mathrm{r}_a}{\mathrm{h}_a}},\sqrt{\frac{\mathrm{r_b}}{\mathrm{h_b}}}\right)<\sqrt{\frac{2\mathrm{R}-\mathrm{r}+\mathrm{AI}+\mathrm{BI}+\mathrm{CI}}{\mathrm{r}}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{For } 0 < x < y, \text{we have} : I(x, y) = \frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}} < \frac{x+y}{2}$$

$$\text{Reference} :$$
$$\left(\begin{array}{c}\text{Article titled "Geometric Inequalities with Classical Means" by Bogdan Fustei,}\\ \text{published at : www.ssmrmh.ro}\end{array}\right)$$

$$\therefore \sum_{cyc} I\left(\sqrt{\frac{r_a}{h_a}}, \sqrt{\frac{r_b}{h_b}}\right) < \sum_{cyc} \frac{\sqrt{\frac{r_a}{h_a}} + \sqrt{\frac{r_b}{h_b}}}{2} = \sum_{cyc} \sqrt{\frac{s\tan\frac{A}{2}.4R\cos\frac{A}{2}\sin\frac{A}{2}}{2rs}}$$

$$= \sqrt{\frac{2R}{r}}.\sum_{cyc}\sin\frac{A}{2} \Rightarrow \left(\sum_{cyc} I\left(\sqrt{\frac{r_a}{h_a}}, \sqrt{\frac{r_b}{h_b}}\right)\right)^2 < \frac{2R}{r}.\left(\sum_{cyc}\sin^2\frac{A}{2} + 2\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2}\right)$$

$$= \frac{2R}{r}.\frac{2R-r}{2R} + \frac{4R}{r}.\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2} \overset{?}{=} \frac{2R - r + AI + BI + CI}{r}$$

$$\Leftrightarrow 16R^2\left(\sum_{cyc}\sin^2\frac{A}{2}\sin^2\frac{B}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\sum_{cyc}\sin\frac{A}{2}\right) \overset{?}{=} \sum_{cyc} AI^2 + 2\sum_{cyc} AI.BI$$

$$\Leftrightarrow 16R^2.\frac{r^2}{16R^2}\sum_{cyc}\csc^2\frac{A}{2} + 16R^2.\frac{r}{2R}.\sum_{cyc}\sin\frac{A}{2} \overset{?}{=}$$

$$r^2.\sum_{cyc}\csc^2\frac{A}{2} + \frac{2r^2}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}.\sum_{cyc}\sin\frac{A}{2} \Leftrightarrow 8Rr.\sum_{cyc}\sin\frac{A}{2} \overset{?}{=} \frac{2r^2.4R}{r}.\sum_{cyc}\sin\frac{A}{2}$$

$$\rightarrow \text{true} \therefore \sum_{cyc} I\left(\sqrt{\frac{r_a}{h_a}}, \sqrt{\frac{r_b}{h_b}}\right) < \sqrt{\frac{2R - r + AI + BI + CI}{r}} \ \forall \Delta ABC \text{ (QED)}$$

3703. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(sin^2(A) + cos^2\left(\frac{A}{2}\right)\right) \le \frac{17}{4} + \frac{r^2}{R^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\left(sin^2(A) + cos^2\left(\frac{A}{2}\right)\right) = \sum_{cyc}\left(sin^2(A)\right) + \sum_{cyc}\left(cos^2\left(\frac{A}{2}\right)\right) =$$

$$\frac{s^2 - r^2 - 4Rr}{2R^2} + \left(2 + \frac{r}{2R}\right) \overset{Gerretsen}{\leq}$$

$$\frac{4R^2 + 4Rr + 3r^2 - r^2 - 4Rr}{2R^2} + 2 + \frac{r}{2R} =$$

$$\frac{4R^2+4R^2+2r^2+Rr}{2R^2}=\frac{8R^2+2r^2+Rr}{2R^2}=$$

$$4+\frac{r}{2R}+\left(\frac{r}{R}\right)^2\overset{Euler}{\leq}4+\frac{1}{4}+\left(\frac{r}{R}\right)^2=\frac{17}{4}+\frac{r^2}{R^2}$$

Equality holds if the triangle is an equilateral one.

3704. ***In ΔABC holds:***

$$\frac{72r^2}{R}\leq\sum_{cyc}\frac{h_a}{sin^2\left(\frac{A}{2}\right)}\leq\frac{8(R^4-7r^4)}{Rr^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{h_a}{sin^2\left(\frac{A}{2}\right)}\overset{Chebyshev}{\geq}\frac{1}{3}\sum_{cyc}h_a\cdot\sum_{cyc}\frac{1}{sin^2\left(\frac{A}{2}\right)}=$$

$$\frac{1}{3}\sum_{cyc}h_a\cdot\sum_{cyc}\frac{2Rh_a}{r.r_a}=\frac{1}{3}\frac{p^2+r^2+4Rr}{2R}\cdot\frac{2R}{r}\cdot\sum_{cyc}\frac{h_a}{r_a}\overset{Euler}{\geq}$$

$$\frac{1}{3}\cdot\frac{27r^2+r^2+8r^2}{r}\cdot\sum_{cyc}\frac{h_a}{r_a}\overset{\substack{Chebyshev\\Euler}}{\geq}\frac{1}{3}\cdot\frac{36r^2}{r}\cdot\frac{1}{3}\sum_{cyc}h_a\cdot\sum_{cyc}\frac{1}{r_a}=$$

$$4r.\frac{p^2+r^2+4Rr}{2R}\cdot\frac{1}{r}\overset{Euler}{\geq}2.\frac{36r^2}{R}=\frac{72r^2}{R}$$

Here the conditions of Chebyshev's theorem

are taken into account : $a\leq b\leq c$; $h_a\geq h_b\geq h_c$

$$\sin\left(\frac{A}{2}\right)\leq\sin\left(\frac{B}{2}\right)\leq\sin\left(\frac{C}{2}\right);\ \frac{1}{r_a}\geq\frac{1}{r_b}\geq\frac{1}{r_c};\ \frac{1}{\sin\left(\frac{A}{2}\right)}\geq\frac{1}{\sin\left(\frac{B}{2}\right)}\geq\frac{1}{\sin\left(\frac{C}{2}\right)}$$

$$\sum_{cyc}\frac{h_a}{sin^2\left(\frac{A}{2}\right)}=\sum_{cyc}\frac{h_a.bc}{(p-b)(p-c)}=\sum_{cyc}\frac{h_a.2F}{\sin(A).(p-b)(p-c)}=$$

$$\sum_{cyc}\frac{h_a.2F.2R}{a(p-b)(p-c)}=\sum_{cyc}\frac{h_a.2F.2R.h_a}{2F(p-b)(p-c)}=\sum_{cyc}\frac{2R.h_a^2}{(p-b)(p-c)}\leq$$

$$2R\sum_{cyc}\frac{p(p-a)}{(p-b)(p-c)}=2Rp\sum_{cyc}\frac{(p-a)}{(p-b)(p-c)}=2Rp\frac{\sum_{cyc}(p-a)^2}{\prod_{cyc}(p-a)}=$$

$$2Rp\frac{(-p^2)+\sum_{cyc}a^2}{\prod_{cyc}(p-a)}=\frac{2Rp^2}{F^2}\left((-p^2)+\sum_{cyc}a^2\right)=$$

$$\frac{2Rp^2}{p^2r^2}\left((-p^2)+2p^2-2r^2-8Rr\right)=\frac{2R}{r^2}(p^2-2r^2-8Rr)\overset{Gerretsen}{\leq}$$

$$\frac{2R}{r^2}(4R^2+4Rr+3r^2-2r^2-8Rr)=\frac{2R}{r^2}(4R^2-4Rr+r^2)=$$

$$\frac{2R^2}{Rr^2}(4R^2-4Rr+r^2)=\frac{8R^4-8R^3r+2R^2r^2}{Rr^2}$$

Showing that the inequality is true

$$\frac{8R^4-8R^3r+2R^2r^2}{Rr^2}\leq\frac{8(R^4-7r^4)}{Rr^2}$$

$$8R^4-8R^3r+2R^2r^2\leq 8(R^4-7r^4)$$

$$8R^3r+2R^2r^2-56r^4\geq 0$$

$$4R^3r+R^2r^2-28r^4\geq 0$$

$$4\left(\frac{R}{r}\right)^3+\left(\frac{R}{r}\right)^2-28\geq 0$$

$$4\left(\left(\frac{R}{r}\right)^3-8\right)+\left(\frac{R}{r}\right)^2+4\geq 0$$

$$4\left(\frac{R}{r}-2\right)\left(\left(\frac{R}{r}\right)^2+2\left(\frac{R}{r}\right)+4\right)+\left(\frac{R}{r}\right)^2+4\geq 0$$

$$\frac{R}{r}-2\geq 0\ \rightarrow\ \frac{R}{r}\geq 2\ \rightarrow\ R\geq 2r\ (Euler)\ (true)$$

Equality holds for an equilateral triangle.

3705. In ΔABC the following relationship holds:

$$\sum\frac{bc}{b^2+c^2}\leq\frac{R^2}{4r^2}\sum\frac{a^2}{b^2+c^2}\leq\frac{3R^4}{32r^4}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a^2}{b^2+c^2}\overset{AM-GM}{\leq}\sum\frac{a^2}{2bc}=\frac{1}{2abc}\sum a^3=\frac{2s(s^2-3r^2-6Rr)}{8Rrs}\overset{Gerretsen}{\leq}$$

$$\leq\frac{4R^2-2Rr}{4Rr}=\frac{R}{r}-\frac{1}{2}$$

$$then\ \frac{R^2}{4r^2}\sum\frac{a^2}{b^2+c^2}\le\frac{R^2}{4r^2}\left(\frac{R}{r}-\frac{1}{2}\right)=\frac{R^3}{4r^3}-\frac{R^2}{8r^2}$$

We need to show:

$$\frac{R^2}{4r^2}\sum\frac{a^2}{b^2+c^2}\le\frac{3R^4}{32r^4}\quad or$$

$$\frac{R^3}{4r^3}-\frac{R^2}{8r^2}\le\frac{3R^4}{32r^4}\ or\ 3x^2-8x+4\overset{\frac{R}{r}=x\ge 2}{\ge}0\ or(x-2)(3x-2)\ge 0\ \ true\ (A)$$

$$\sum\frac{bc}{b^2+c^2}\overset{AM-GM}{\le}\frac{1}{2}\sum\frac{bc}{bc}=\frac{3}{2}\ (B)$$

$$\frac{R^2}{4r^2}\sum\frac{a^2}{b^2+c^2}\overset{Nesbitt}{\ge}\frac{3}{2}\frac{R^2}{4r^2}\overset{Euler}{\ge}\frac{3}{2}\ (C)$$

$$From\ (A),(B),(C) we\ have:\sum\frac{bc}{b^2+c^2}\le\frac{R^2}{4r^2}\sum\frac{a^2}{b^2+c^2}\le\frac{3R^4}{32r^4}$$

Equality holds for an equilateral triangle.

3706. In ΔABC the following relationship holds:

$$\sqrt[2026]{\sin A}+\sqrt[2026]{\sin B}+\sqrt[2026]{\sin C}\le\sqrt[2026]{\cos\frac{A}{2}}+\sqrt[2026]{\cos\frac{B}{2}}+\sqrt[2026]{\cos\frac{C}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sin A+\sin B=2\sin\frac{A+B}{2}.\cos\frac{A-B}{2}\overset{A+B+C=\pi\ \&\ \cos\frac{A-B}{2}\le 1}{\le}2\cos\frac{C}{2}$$

$$\sqrt[2026]{\sin A}+\sqrt[2026]{\sin B}\overset{CBS}{\le}2\sqrt[2026]{\left(\frac{\sin A+\sin B}{2}\right)}\le 2\sqrt[2026]{\left(\frac{2\cos\frac{C}{2}}{2}\right)}=$$

$$=2\sqrt[2026]{\cos\frac{A}{2}}\ then\ \sqrt[2026]{\cos\frac{A}{2}}\ge\frac{1}{2}\left(\sqrt[2026]{\sin A}+\sqrt[2026]{\sin B}\right)$$

$$\sqrt[2026]{\cos\frac{A}{2}}+\sqrt[2026]{\cos\frac{B}{2}}+\sqrt[2026]{\cos\frac{C}{2}}\ge\frac{1}{2}\sum\left(\sqrt[2026]{\sin A}+\sqrt[2026]{\sin B}\right)=$$

$$=\sqrt[2026]{\sin A}+\sqrt[2026]{\sin B}+\sqrt[2026]{\sin C}$$

Equality holds for an equilateral triangle.

3707. If $n\ge 3$ then in ΔABC the following relationship holds:

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}+\frac{R^n}{r^{n-1}}\ge 2^n r+\frac{\sqrt[3]{abc}}{3}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\overset{CBS}{\le}\sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)}\overset{Steining}{\le}\sqrt{\frac{3}{4r^2}}=\frac{\sqrt{3}}{2r}\ (1)$$

$$abc=4Rrs\overset{Euler\ \&\ Mitrinovic}{\le}4.R.\frac{R}{2}.\frac{3\sqrt{3}R}{2}=\left(\sqrt{3}R\right)^3\ (2)$$

We need to show:

$$\frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}+\frac{R^n}{r^{n-1}}\ge 2^n r+\frac{\sqrt[3]{abc}}{3}$$

$$\frac{1}{\frac{\sqrt{3}}{2r}}+\frac{R^n}{r^{n-1}}\ge 2^n r+\frac{\sqrt{3}R}{3}\ or\ \frac{2r}{\sqrt{3}}+\frac{R^n}{r^{n-1}}\ge 2^n r+\frac{\sqrt{3}R}{3}$$

$$\frac{2}{\sqrt{3}}+\left(\frac{R}{r}\right)^n\ge 2^n+\frac{R}{r\sqrt{3}}\ or\ \sqrt{3}(x^n-2^n)-(x-2)\overset{\frac{R}{r}=x\ge 2\ Euler}{\ge}0$$

$$\sqrt{3}(x-2)(x^{n-1}+2x^{n-2}+4.x^{n-3}\dots+2^{n-1})-(x-2)\ge 0$$

$$(x-2)\left(\sqrt{3}(x^{n-1}+2x^{n-2}+4.x^{n-3}\dots+2^{n-1})-1\right)\ge 0$$

$$true\ \ as\ x\ge 2\ and\ for\ n\ge 3,(x^{n-1}+2x^{n-2}+4.x^{n-3}\dots+2^{n-1})-1>0$$

Equality holds for an equilateral triangle.

3708.

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sqrt{m_b m_c}.\left(m_b^4(m_a+m_b)^2+m_c^4(m_c+m_a)^2\right)}{m_b^2.\sqrt{m_a m_b}+m_c^2.\sqrt{m_c m_a}}\geq 3\cdot(18r^2)^2$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A', B', C', x', y', z' > 0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\underbrace{\overset{\text{Walter Janous}}{\geq}}_{(1)}\sqrt{3\sum_{cyc}A'B'}$$

$$\text{Now, }\sum_{cyc}\frac{\sqrt{m_b m_c}.\left(m_b^4(m_a+m_b)^2+m_c^4(m_c+m_a)^2\right)}{m_b^2.\sqrt{m_a m_b}+m_c^2.\sqrt{m_c m_a}}$$

$$=\sum_{cyc}\frac{\frac{\sqrt{m_b m_c}}{m_a^2}.\left(\frac{m_c^2 m_a^2}{m_b^2}(m_c+m_a)^2+\frac{m_a^2 m_b^2}{m_c^2}(m_a+m_b)^2\right)}{\frac{\sqrt{m_c m_a}}{m_b^2}+\frac{\sqrt{m_a m_b}}{m_c^2}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{matrix}x'=\frac{\sqrt{m_b m_c}}{m_a^2}, y'=\frac{\sqrt{m_c m_a}}{m_b^2}, z'=\frac{\sqrt{m_a m_b}}{m_c^2},\\ A'=\frac{m_b^2 m_c^2}{m_a^2}(m_b+m_c)^2, B'=\frac{m_c^2 m_a^2}{m_b^2}(m_c+m_a)^2, C'=\frac{m_a^2 m_b^2}{m_c^2}(m_a+m_b)^2\end{matrix}\right)\overset{\text{via }(1)}{\geq}$$

$$\sqrt{3\sum_{cyc}\left(\frac{m_b^2 m_c^2}{m_a^2}(m_b+m_c)^2.\frac{m_c^2 m_a^2}{m_b^2}(m_c+m_a)^2\right)}\overset{\text{AM-GM}}{\geq}\sqrt{3\sum_{cyc}m_c^4.\left((4m_b m_c).(4m_c m_a)\right)}$$

$$\overset{\text{AM-GM}}{\geq}3.\sqrt{\sqrt[3]{4^6(m_a m_b m_c)^8}}\overset{\text{Lascu + AM-GM}}{\geq}12.\left(\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}\right)^{\frac{4}{3}}$$

$$=12.(rs^2)^{\frac{4}{3}}\overset{\text{Mitrinovic}}{\geq}12.(27r^3)^{\frac{4}{3}}=12(3r)^4=3.(18r^2)^2\text{ and so,}$$

$$\sum_{cyc}\frac{\sqrt{m_b m_c}.\left(m_b^4(m_a+m_b)^2+m_c^4(m_c+m_a)^2\right)}{m_b^2.\sqrt{m_a m_b}+m_c^2.\sqrt{m_c m_a}}\geq 3.(18r^2)^2\ \forall\,\Delta\text{ ABC},$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

3709. In ΔABC the following relationship holds:

$$\sum\left(\frac{h_a}{h_bh_c}\right)^3 \ge \frac{1}{9r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ x=\frac{r}{h_a}, y=\frac{r}{h_b}, z=\frac{r}{h_c}\ then\ x+y+z=1\ as$$

$$\sum\frac{1}{h_a}=\frac{1}{r}, \frac{h_a}{h_bh_c}=\frac{\frac{1}{h_bh_c}}{\frac{1}{h_a}}=\frac{1}{r}\frac{\frac{r}{h_b}\cdot\frac{r}{h_c}}{\frac{r}{h_a}}=\frac{1}{r}\cdot\frac{yz}{x}$$

$$\sum\left(\frac{h_a}{h_bh_c}\right)^3=\frac{1}{r^3}\sum\left(\frac{yz}{x}\right)^3=\frac{1}{r^3}\frac{y^6z^6+z^6x^6+x^6y^6}{x^3y^3z^3}\overset{\forall a,b,c>0\ \sum a^2\ge\sum ab}{\ge}$$

$$\ge\frac{1}{r^3}\frac{x^3y^3z^3(x^3+y^3+z^3)}{x^3y^3z^3}\overset{CBS}{\ge}\frac{1}{r^3}\frac{1}{9}(x+y+z)^3=\frac{1}{9r^3}$$

Equality holds for an equilateral triangle.

3710. In acute ΔABC the following relationship holds:

$$\frac{a^2+b^2}{cosC}+\frac{b^2+c^2}{cosA}+\frac{c^2+a^2}{cosB}\ge 36R^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{a^2+b^2}{cosC}+\frac{b^2+c^2}{cosA}+\frac{c^2+a^2}{cosB}=\sum_{cyc}\frac{a^2+b^2}{cosC}\overset{AM-GM}{\ge}$$

$$\ge 2\sum_{cyc}\frac{ab}{cosC}=2\sum_{cyc}\frac{\frac{2F}{sinC}}{cosC}=8F\sum_{cyc}\frac{1}{2sinCcosC}=8F\sum_{cyc}\frac{1}{sin2C}\ge$$

$$\overset{BERGSTROM}{\ge} 8F\cdot\frac{(1+1+1)^2}{sin2A+sin2B+sin2C}=\frac{72F}{4sinAsinBsinC}=$$

$$=\frac{18F}{\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}}=\frac{18F\cdot 8R^3}{4RF}=36R^2$$

Equality holds for an equilateral triangle.

3711. In ΔABC the following relationship holds:

$$\frac{a}{\sqrt{(b+c)^2-a^2}}+\frac{b}{\sqrt{(c+a)^2-b^2}}+\frac{c}{\sqrt{(a+b)^2-c^2}}\ge\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{a}{\sqrt{(b+c)^2-a^2}}+\frac{b}{\sqrt{(c+a)^2-b^2}}+\frac{c}{\sqrt{(a+b)^2-c^2}}=$$

$$=\sum_{cyc}\frac{a}{\sqrt{(b+c)^2-a^2}}=\sum_{cyc}\frac{a}{\sqrt{(b+c-a)(b+c+a)}}=$$

$$=\sum_{cyc}\frac{a}{\sqrt{2s(2s-2a)}}=\frac{1}{2}\sum_{cyc}\frac{a}{\sqrt{s(s-a)}}\overset{AM-GM}{\ge}\frac{1}{2}\cdot 3\sqrt[3]{\frac{abc}{s\cdot\sqrt{s(s-a)(s-b)(s-c)}}}=$$

$$=\frac{3}{2}\cdot\sqrt[3]{\frac{4RF}{s\cdot F}}=\frac{3}{2}\cdot\sqrt[3]{\frac{4R}{s}}\overset{MITRINOVICI}{\ge}\frac{3}{2}\cdot\sqrt[3]{\frac{4\cdot\frac{2}{3\sqrt{3}}\cdot s}{s}}=\frac{3}{2}\sqrt[3]{\frac{8}{3\sqrt{3}}}=\frac{3}{2}\cdot\frac{2}{\sqrt{3}}=\sqrt{3}$$

Equality holds for an equilateral triangle.

3712. In acute ΔABC the following relationship holds:

$$\tan^{2026}A+\tan^{2026}B+\tan^{2026}C\ge\left(\cot^{2026}\frac{A}{2}+\cot^{2026}\frac{B}{2}+\cot^{2026}\frac{C}{2}\right)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\tan A+\tan B=\frac{sinA}{cosA}+\frac{sinB}{cosB}=\frac{\sin(A+B)}{cosAcosB}\overset{A+B+C=\pi}{=}$$

$$=\frac{sinC}{cosAcosB}=\frac{2sinC}{\cos(A+B)+\cos(A-B)}\overset{A+B+C=\pi \& \cos(A-B)\le 1}{=}$$

$$=\frac{2sinC}{1-\cos C}=\frac{4\sin\frac{C}{2}\cos\frac{c}{2}}{2\sin^2\left(\frac{C}{2}\right)}=2\cot\frac{C}{2}$$

$$\tan^{2026}A+\tan^{2026}B\overset{CBS}{\ge}\frac{2}{2^{2026}}(\tan A+\tan B)^{2026}\ge$$

$$\ge\frac{2}{2^{2026}}\left(2\cot\frac{C}{2}\right)^{2026}=2\cot^{2026}\frac{C}{2}\text{ or }\cot^{2026}\frac{C}{2}\le\frac{1}{2}(\tan^{2026}A+\tan^{2026}B)$$

$$\left(\cot^{2026}\frac{A}{2}+\cot^{2026}\frac{B}{2}+\cot^{2026}\frac{C}{2}\right)=$$

$$=\sum\cot^{2026}\frac{C}{2}\le\frac{1}{2}\sum(\tan^{2026}A+\tan^{2026}B)\ =\tan^{2026}A+\tan^{2026}B+\tan^{2026}C$$

Equality holds for an equilateral triangle.

3713. **In ΔABC the following relationship holds:**

$$\frac{a\cos(A)+b\cos(B)+c\cos(C)}{a\sin(A)+b\sin(B)+c\sin(C)}\le\frac{\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$a\cos(A)+b\cos(B)+c\cos(C)=\sum_{cyc}a\cos(A)=2R\sum_{cyc}sinAcosA=$$

$$=R\sum_{cyc}sin2A=4RsinAsinBsinC=\frac{2F}{R}=\frac{2rs}{R}$$

$$a\sin(A)+b\sin(B)+c\sin(C)=\sum_{cyc}a\cdot\frac{a}{2R}=\frac{a^2+b^2+c^2}{2R}$$

$$\frac{a\cos(A)+b\cos(B)+c\cos(C)}{a\sin(A)+b\sin(B)+c\sin(C)}=\frac{\frac{2rs}{R}}{\frac{a^2+b^2+c^2}{2R}}=$$

$$=\frac{4rs}{a^2+b^2+c^2}\overset{IONESCU-WEITZENBOCK}{\lesseqgtr}\le\frac{4F}{4\sqrt{3}F}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

$$Equality\ \ holds\ \ for: a=b=c.$$

3714. In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{1}{3\tan^2\frac{A}{2}+1} \ge \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{3\tan^2\frac{A}{2}+1} = \sum_{cyc} \frac{s^2(3r_b^2+s^2)(3r_c^2+s^2)}{(3r_a^2+s^2)(3r_b^2+s^2)(3r_c^2+s^2)}$$

$$= \frac{9s^2\sum_{cyc} r_b^2r_c^2 + 6s^4\sum_{cyc} r_a^2 + 3s^6}{27r_a^2r_b^2r_c^2 + 9s^2\sum_{cyc} r_b^2r_c^2 + 3s^4\sum_{cyc} r_a^2 + s^6} \overset{?}{\ge} \frac{3}{2}$$

$$\Leftrightarrow 3s^4\sum_{cyc} r_a^2 + 3s^6 \overset{?}{\ge} 9s^2\sum_{cyc} r_b^2r_c^2 + 81r^2s^4$$

$$\Leftrightarrow s^2((4R+r)^2-2s^2)+s^4 \overset{?}{\ge} 3\left(s^4-2s^2r(4R+r)\right)+27r^2s^2$$

$$\Leftrightarrow s^2 \overset{?}{\le} 4R^2+8Rr-5r^2 \Leftrightarrow (s^2-4R^2-4Rr-3r^2)-4r(R-2r) \overset{?}{\le} 0 \to \text{true}$$

$$\because s^2-4R^2-4Rr-3r^2 \overset{\text{Gerretsen}}{\le} 0 \text{ and } 4r(R-2r) \overset{\text{Euler}}{\ge} 0$$

$$\therefore \sum_{cyc} \frac{1}{3\tan^2\frac{A}{2}+1} \ge \frac{3}{2}\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3715. In any acute Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{1}{3\cot^2 A+1} \ge \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{3\tan^2\frac{A}{2}+1} = \sum_{cyc} \frac{s^2(3r_b^2+s^2)(3r_c^2+s^2)}{(3r_a^2+s^2)(3r_b^2+s^2)(3r_c^2+s^2)}$$

$$= \frac{9s^2\sum_{cyc} r_b^2r_c^2 + 6s^4\sum_{cyc} r_a^2 + 3s^6}{27r_a^2r_b^2r_c^2 + 9s^2\sum_{cyc} r_b^2r_c^2 + 3s^4\sum_{cyc} r_a^2 + s^6} \overset{?}{\ge} \frac{3}{2}$$

$$\Leftrightarrow 3s^4\sum_{cyc} r_a^2 + 3s^6 \overset{?}{\ge} 9s^2\sum_{cyc} r_b^2r_c^2 + 81r^2s^4$$

$$\Leftrightarrow s^2\left((4R+r)^2-2s^2\right)+s^4 \overset{?}{\geq} 3\left(s^4-2s^2r(4R+r)\right)+27r^2s^2$$

$$\Leftrightarrow s^2 \overset{?}{\leq} 4R^2+8Rr-5r^2 \Leftrightarrow (s^2-4R^2-4Rr-3r^2)-4r(R-2r) \overset{?}{\leq} 0 \to \text{true}$$

$$\because s^2-4R^2-4Rr-3r^2 \overset{\text{Gerretsen}}{\leq} 0 \text{ and } 4r(R-2r) \overset{\text{Euler}}{\geq} 0$$

$$\therefore \sum_{\text{cyc}} \frac{1}{3\tan^2\frac{A}{2}+1} \geq \frac{3}{2}\ \forall\,\Delta\,ABC, \text{and implementing it on a triangle with angles :}$$

$$(\pi-2A),(\pi-2B),(\pi-2C), \text{we get : } \sum_{\text{cyc}} \frac{1}{3\tan^2\frac{\pi-2A}{2}+1} \geq \frac{3}{2}$$

$$\Rightarrow \sum_{\text{cyc}} \frac{1}{3\cot^2 A+1} \geq \frac{3}{2} \text{ and since the latter triangle is an acute triangle,}$$

$$\therefore \sum_{\text{cyc}} \frac{1}{3\cot^2 A+1} \geq \frac{3}{2}\ \forall \text{ acute } \Delta\,ABC, ''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3716. In any acute Δ ABC the following relationship holds :

$$\sec^2 A+\sec^2 B+\sec^2 C \geq \csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}} \sec^2 A \overset{?}{\geq} \sum_{\text{cyc}} \csc^2\frac{A}{2} \Leftrightarrow \sum_{\text{cyc}} \tan^2 A \overset{?}{\geq} \sum_{\text{cyc}} \cot^2\frac{A}{2}$$

$$\Leftrightarrow \left(\prod_{\text{vyc}} \tan A\right)^2 - 2\left(\prod_{\text{vyc}} \tan A\right)\left(\sum_{\text{cyc}} \cot A\right) \overset{?}{\geq} \frac{s^2}{r^2s^4}\left(s^4-2s^2r(4R+r)\right)$$

$$\Leftrightarrow \left(\frac{2rs}{s^2-4R^2-4Rr-r^2}\right)^2 - 2\left(\frac{2rs}{s^2-4R^2-4Rr-r^2}\right)\left(\frac{2(s^2-4Rr-r^2)}{4rs}\right)$$

$$\overset{?}{\geq} \frac{s^2-8Rr-2r^2}{r^2}$$

$$\Leftrightarrow -s^6+(8R^2+16Rr+2r^2)s^4-(16R^4+96R^3r+96R^2r^2+24Rr^3-3r^4)s^2+$$

$$r(128R^5+288R^4r+224R^3r^2+72R^2r^3+8Rr^4) \boxed{\overset{?}{\underset{(1)}{\geq}}} 0 \text{ and } \because P =$$

$$-(s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3)(s^2-4R^2-4Rr-r^2) \overset{\text{Double-Rouche}}{\geq} 0$$

$$(\because \Delta\,ABC \to \text{acute} \Rightarrow s > 2R+r) \therefore \text{ in order to prove (1), it suffices to prove :}$$

$$\text{LHS of (1)} \overset{?}{\geq} P \Leftrightarrow -(8R-3r)s^4+(64R^3+28R^2r+2r^3)s^2-$$

$$\left(128R^5+160R^4r+80R^3r^2+28R^2r^3+8Rr^4+r^5\right)\boxed{\underset{②}{\overset{?}{\geq}}}0 \text{ and } \because Q=$$
$$-(8R-3r)(s^2-4R^2-4Rr-3r^2)(s^2-4R^2-4Rr-r^2)\overset{\text{Gerretsen}}{\geq}0$$
$$\therefore \text{ in order to prove ②, it suffices to prove : LHS of ②} \overset{?}{\geq} Q \Leftrightarrow$$
$$24R^4+40R^3r+2R^2r^2-16Rr^3-5r^4\boxed{\underset{③}{\overset{?}{\geq}}}(6R^2+4Rr-7r^2)s^2 \text{ and now,}$$
$$\text{RHS of ③} \overset{\text{Rouche}}{\leq}(6R^2+4Rr-7r^2)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)$$
$$\overset{?}{\leq}\text{LHS of ③} \Leftrightarrow 2(R-2r)(6R^3-2R^2r-13Rr^2+3r^3)\underset{(*)}{\overset{?}{\geq}}$$
$$2(R-2r).\sqrt{R^2-2Rr}.(6R^2+4Rr-7r^2) \text{ and } \because R-2r\overset{\text{Euler}}{\geq}0 \text{ and}$$
$$6R^3-2R^2r-13Rr^2+3r^3=(R-2r)(6R^2+10Rr+7r^2)+17r^3\overset{\text{Euler}}{\geq}17r^3>0$$
$$\therefore \text{ in order to prove } (*), \text{ it suffices to prove :}$$
$$(6R^3-2R^2r-13Rr^2+3r^3)^2\overset{?}{>}(R^2-2Rr)(6R^2+4Rr-7r^2)^2$$
$$\Leftrightarrow r^2(12R^4+6R^3r+2R^2r(R-2r)+20Rr^3+9r^4)\overset{?}{>}0\to \text{true}\Rightarrow(*)\Rightarrow ③\Rightarrow ②$$
$$\Rightarrow ① \text{ is true } \therefore \sec^2A+\sec^2B+\sec^2C\geq\csc^2\frac{A}{2}+\csc^2\frac{B}{2}+\csc^2\frac{C}{2}$$
$$\forall \text{ acute } \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3717. In ΔABC the following relationship holds:

$$\sum \sin A(\cot B+\cot C)\geq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Sarkhan Adgozalov-Georgia

$$\sum \sin A(\cot B+\cot C)=\sum \sin A\left(\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}\right)=$$
$$=\sum\left(\frac{\sin A}{\sin B}\cdot\cos B+\frac{\sin A}{\sin C}\cdot\cos C\right)=\sum\left(\frac{a}{b}\cdot\frac{a^2+c^2-b^2}{2ac}+\frac{a}{c}\cdot\frac{a^2+b^2-c^2}{2ab}\right)=$$
$$\sum\left(\frac{a^2+c^2-b^2}{2bc}+\frac{a^2+b^2-c^2}{2bc}\right)=\sum\frac{a^2}{bc}\geq\frac{(a+b+c)^2}{ab+bc+ca}\geq 3$$

$$\textit{Equality holds for}: a=b=c.$$

3718. In *any acute* Δ ABC the *following relationship holds* :

$$\sqrt{\cos A}+\sqrt{\cos B}+\sqrt{\cos C}\le\sqrt{\sin\frac{A}{2}}+\sqrt{\sin\frac{B}{2}}+\sqrt{\sin\frac{C}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{\cos A}+\sqrt{\cos B}+\sqrt{\cos C}=\frac{1}{2}\sum_{cyc}\left(\sqrt{\cos B}+\sqrt{\cos C}\right)\overset{CBS}{\le}$$

$$\le\frac{1}{2}\cdot\sum_{cyc}\left(\sqrt{2(\cos B+\cos C)}\right)=\frac{1}{\sqrt{2}}\cdot\sum_{cyc}\left(\sqrt{2\cos\frac{B+C}{2}\cdot\cos\frac{B-C}{2}}\right)$$

$$=\sum_{cyc}\sqrt{\sin\frac{A}{2}\cdot\cos\frac{B-C}{2}}\le\sum_{cyc}\sqrt{\sin\frac{A}{2}}\left(\because 0<\cos\frac{B-C}{2}\le 1 \text{ and analogs}\right)$$

$$\text{and so,}\sum_{cyc}\sqrt{\cos A}\le\sum_{cyc}\sqrt{\sin\frac{A}{2}}\ \forall\text{ acute }\Delta\text{ ABC,}$$

$$''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

3719. In ΔABC the following relationship holds:

$$\sum_{cyc}a^2\cos\frac{B}{2}\cos\frac{C}{2}\ge 27r^2$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}a^2\cos\frac{B}{2}\cos\frac{C}{2}=\sum_{cyc}a^2\sqrt{\frac{s(s-b)}{ac}}\cdot\sqrt{\frac{s(s-c)}{ab}}=$$

$$=\sum_{cyc}as\sqrt{\frac{(s-b)(s-c)}{bc}}=s\sum_{cyc}a\sin\frac{A}{2}\overset{CEBYSHEV}{\ge}$$

$$\geq \frac{s}{3}\sum_{cyc} a \sum_{cyc} sin\frac{A}{2} \overset{JENSEN}{\geq} \frac{s}{3}\cdot 2s\cdot 3sin\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)=$$

$$= 2s^2 sin\frac{A+B+C}{6} = 2s^2 sin\frac{\pi}{6} \overset{MITRINOVICI}{\geq} 2\cdot\left(3\sqrt{3}r\right)^2\cdot\frac{1}{2} = 27r^2$$

Equality holds for an equilateral triangle.

3720. In ΔABC the following relationship holds:

$$\frac{sin^2A}{1+cosA}+\frac{sin^2B}{1+cosB}+\frac{sin^2C}{1+cosC}\geq\frac{3}{2}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Daniel Sitaru

$$\sum_{cyc}\frac{sin^2A}{1+cosA}=\sum_{cyc}\frac{1-cos^2A}{1+cosA}=\sum_{cyc}(1-cosA)=$$

$$=3-\sum_{cyc}cosA \overset{JENSEN}{\geq} 3-3cos\left(\frac{A+B+C}{3}\right)=3-3cos\frac{\pi}{3}=3-\frac{3}{2}=\frac{3}{2}$$

Equality holds for an equilateral triangle.

3721. In ΔABC the following relationship holds:

$$\left(\frac{r_a}{w_bw_c}\right)^2+\left(\frac{r_b}{w_cw_a}\right)^2+\left(\frac{r_c}{w_aw_b}\right)^2\geq\left(\frac{3}{s}\right)^2$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$Lemma\ 1$: $\dfrac{r_ar_br_c}{w_aw_bw_c}=\dfrac{(a+b)(b+c)(a+c)}{8abc}\geq\dfrac{8abc}{8abc}=1$

$Proof$: $r_a=\dfrac{F}{s-a}$, $r_b=\dfrac{F}{s-b}$, $r_c=\dfrac{F}{s-c}$

$$w_a=\frac{2}{b+c}\sqrt{bcs(s-a)},\ w_b=\frac{2}{a+c}\sqrt{acs(s-b)},\ w_c=\frac{2}{a+b}\sqrt{abs(s-c)}$$

$$\prod w_a=\frac{8sabc}{(a+b)(b+c)(c+a)}\cdot\sqrt{s(s-a)(s-b)(s-c)}$$

$$=\frac{8abc}{(a+b)(b+c)(c+a)}\cdot Fs\ldots(1).$$

$$\prod r_a=\frac{F^3}{(s-a)(s-b)(s-c)}=Fs\ldots(2).$$

$$(1)\wedge(2)\Rightarrow\prod w_a=\frac{8abc}{(a+b)(b+c)(c+a)}\cdot\prod r_a\Rightarrow\frac{\prod r_a}{\prod w_a}$$

$$=\frac{(a+b)(b+c)(c+a)}{8abc}$$

Lemma 2: $w_a^2\le s(s-a)$

Proof: $w_a^2=\frac{4bc}{(b+c)^2}\cdot s(s-a)\le\frac{4bc}{4bc}\cdot s(s-a)=s(s-a)$

$$LHS=\sum\left(\frac{r_a}{w_bw_c}\right)^2\ge 3\cdot\sqrt[3]{\left(\frac{r_ar_br_c}{(w_aw_bw_c)^2}\right)^2}=3\sqrt[3]{\left(\frac{r_ar_br_c}{w_aw_bw_c}\right)^2\cdot\frac{1}{(w_aw_bw_c)}}\ge$$

$$\ge 3\sqrt[3]{\frac{1}{s^3(s-a)(s-b)(s-c)}}=\frac{3}{s}\sqrt[3]{\frac{27}{27(s-a)(s-b)(s-c)}}\ge\frac{9}{s}\sqrt[3]{\frac{1}{(3s-a-b-c)^3}}=\frac{9}{s^2}$$

Equality holds for $a=b=c$.

3722. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}\left(\cot^4\frac{B}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)+\cot^4\frac{C}{2}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)\right)}{\cot^2\frac{B}{2}.\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}+\cot^2\frac{C}{2}.\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}}\ge 108$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\ A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\underbrace{\overset{\text{Walter Janous}}{\ge}}_{①}\sqrt{3\sum_{cyc}A'B'}$$

$$\text{Now, }\sum_{cyc}\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}\left(\cot^4\frac{B}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)+\cot^4\frac{C}{2}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)\right)}{\cot^2\frac{B}{2}.\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}+\cot^2\frac{C}{2}.\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}}$$

$$=\sum_{cyc}\frac{\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}}{\cot^2\frac{A}{2}}\left(\frac{\cot^2\frac{C}{2}\cot^2\frac{A}{2}}{\cot^2\frac{B}{2}}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)^2+\frac{\cot^2\frac{A}{2}\cot^2\frac{B}{2}}{\cot^2\frac{C}{2}}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)^2\right)}{\frac{\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}}{\cot^2\frac{B}{2}}+\frac{\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}}{\cot^2\frac{C}{2}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{array}{c}x'=\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}}{\cot^2\frac{A}{2}},y'=\frac{\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}}{\cot^2\frac{B}{2}},z'=\frac{\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}}{\cot^2\frac{C}{2}},A'=\frac{\cot^2\frac{B}{2}\cot^2\frac{C}{2}}{\cot^2\frac{A}{2}}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)^2,\\ B'=\frac{\cot^2\frac{C}{2}\cot^2\frac{A}{2}}{\cot^2\frac{B}{2}}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)^2,C'=\frac{\cot^2\frac{A}{2}\cot^2\frac{B}{2}}{\cot^2\frac{C}{2}}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)^2\end{array}\right)$$

$$\overset{\text{via }①}{\geq}\sqrt{3\sum_{cyc}\left(\frac{\cot^2\frac{B}{2}\cot^2\frac{C}{2}}{\cot^2\frac{A}{2}}\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)^2.\frac{\cot^2\frac{C}{2}\cot^2\frac{A}{2}}{\cot^2\frac{B}{2}}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)^2\right)}$$

$$\overset{AM-GM}{\geq}\sqrt{3\sum_{cyc}\cot^4\frac{C}{2}.\left(\left(4\cot\frac{B}{2}\cot\frac{C}{2}\right).\left(4\cot\frac{C}{2}\cot\frac{A}{2}\right)\right)}\overset{AM-GM}{\geq}3.\sqrt{\sqrt[3]{4^6\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^8}}$$

$$=12.\left(\frac{s}{r}\right)^{\frac{4}{3}}\overset{\text{Mitrinovic}}{\geq}12.\left(3\sqrt{3}\right)^{\frac{4}{3}}=108\text{ and so,}$$

$$\sum_{cyc}\frac{\sqrt{\cot\frac{B}{2}\cot\frac{C}{2}}\left(\cot^4\frac{B}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)+\cot^4\frac{C}{2}\left(\cot\frac{C}{2}+\cot\frac{A}{2}\right)\right)}{\cot^2\frac{B}{2}.\sqrt{\cot\frac{A}{2}\cot\frac{B}{2}}+\cot^2\frac{C}{2}.\sqrt{\cot\frac{C}{2}\cot\frac{A}{2}}}\geq 108\ \forall\ \Delta\ ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)

3723. In ΔABC the following relationship holds:

$$\frac{a}{\sqrt{b+c-a}}+\frac{b}{\sqrt{a+c-b}}+\frac{c}{\sqrt{a+b-c}}\geq\sqrt{a}+\sqrt{b}+\sqrt{c}$$

Proposed by Gheorghe Crăciun-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a}{\sqrt{b+c-a}}+\frac{b}{\sqrt{a+c-b}}+\frac{c}{\sqrt{a+b-c}}\geq \sqrt{a}+\sqrt{b}+\sqrt{c}$$

$$\sum_{cyc}\frac{a}{\sqrt{b+c-a}}=\sum_{cyc}\frac{a}{\sqrt{2}.\sqrt{p-a}}=\frac{1}{\sqrt{2}}\sum_{cyc}a\sqrt{\frac{r_a}{F}}=$$

$$=\frac{1}{\sqrt{2}}\sum_{cyc}a\sqrt{\frac{r_a}{F}}=\frac{1}{\sqrt{2F}}\sum_{cyc}a\sqrt{r_a}=\frac{1}{\sqrt{2F}}\sum_{cyc}\sqrt{a}\ .\ \sqrt{ar_a}$$

In ΔABC *wlog* $a\leq b\leq c$, $r_a\leq r_b\leq r_c$ *By Chebyshev's inequality* :

$$\frac{1}{\sqrt{2F}}\sum_{cyc}\sqrt{a}\ .\ \sqrt{ar_a}\ \overset{Chebyshev}{\geq}$$

$$\geq\frac{1}{3}.\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt{ar_a}+\sqrt{br_b}+\sqrt{cr_c}\right)\ \overset{AM-GM}{\geq}$$

$$\geq\frac{1}{3}.\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).3\sqrt[3]{\sqrt{(abc)(r_ar_br_c)}}=$$

$$=\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).\left(\sqrt[6]{(4RF).(rS^2)}\right)\ \overset{Euler}{\geq}$$

$$\geq\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right).\left(\sqrt[6]{4.2r.F.(rS^2)}\right)=$$

$$=\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt[6]{8.r^2S^2.F}\right)=$$

$$=\frac{1}{\sqrt{2F}}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\left(\sqrt[6]{8F^3}\right)=\sqrt{a}+\sqrt{b}+\sqrt{c}$$

Equality holds for: $a=b=c$

3724. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{sin^2(A)}{sin^2(B)+sin^2(C)}\leq\frac{2R-r}{2r}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{sin^2(A)}{sin^2(B)+sin^2(C)}=\sum_{cyc}\frac{a^2}{b^2+c^2}\ \overset{AM-GM}{\leq}\ \frac{1}{2}\sum_{cyc}\frac{a^2}{bc}=\sum_{cyc}\frac{a^3}{2abc}=$$

$$=\frac{2(s^3-3sr^2-6sRr)}{2.4Rsr}=\frac{s^2-3r^2-6Rr}{4Rr}\ \overset{Gerretsen}{\leq}$$

$$\leq \frac{4R^2+4Rr+3r^2-3r^2-6Rr}{4Rr}=\frac{4R^2-2Rr}{4Rr}=\frac{2R-r}{2r}$$

$$Equality\ holds\ for:\ A=B=C.$$

3725. In ΔABC the following relationship holds:

$$\frac{w_a^2}{r_b+r_c}+\frac{w_b^2}{r_c+r_a}+\frac{w_c^2}{r_a+r_b}\geq\frac{9r^2}{R}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$Lemma\ 1:\ w_a=\frac{2bc}{b+c}\cdot cos\frac{A}{2}$

$Lemma\ 2:\ r_b+r_c=4R\cdot\cos^2\frac{A}{2}$

$Proof:\ r_b+r_c=s\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=$

$$=\frac{s\cdot\sin\left(\frac{B+C}{2}\right)\cdot\cos\frac{A}{2}}{\prod\cos\frac{A}{2}}=\frac{s\cdot\cos^2\frac{A}{2}}{\frac{s}{4R}}=4R\cdot\cos^2\frac{A}{2}$$

$Lemma\ 3:\ \sum\frac{1}{a^2}\leq\frac{1}{4r^2}$

$Proof:\ r=\frac{F}{s},\ r_a=\frac{F}{s-a}$

$$rr_a=\frac{F^2}{s(s-a)}=(s-b)(s-c)\leq\frac{(s-b+s-c)^2}{4}=\frac{a^2}{4}\ \Rightarrow rr_a\leq\frac{a^2}{4}\ \Rightarrow a^2\geq 4rr_a$$

$$\sum\frac{1}{a^2}\leq\sum\frac{1}{4rr_a}=\frac{1}{4r}\sum\frac{1}{r_a}=\frac{1}{4r^2}$$

$$LHS=\sum\frac{\left(\frac{2bc}{b+c}\right)^2\cdot cos\frac{A}{2}}{4R\cdot\cos^2\frac{A}{2}}=\frac{1}{4R}\cdot\sum\left(\frac{2ac}{a+c}\right)^2=\frac{1}{R}\cdot\sum\left(\frac{ac}{a+c}\right)^2=\frac{1}{R}\cdot\sum\frac{1}{\left(\frac{1}{a}+\frac{1}{c}\right)^2}\geq$$

$$\geq\frac{1}{R}\cdot\frac{9}{\sum\left(\frac{1}{a}+\frac{1}{c}\right)^2}\geq\frac{1}{R}\cdot\frac{9}{2\sum\frac{2}{a^2}}=\frac{1}{R}\cdot\frac{9}{4\sum\frac{1}{a^2}}\geq\frac{1}{R}\cdot\frac{9}{4\cdot\frac{1}{4r^2}}=\frac{9r^2}{R}$$

Equality holds for $a = b = c$.

3726. In any Δ ABC the following relationship holds :

$$2n_a \geq \sqrt{\frac{w_a}{g_a}}\cdot\left(p_a + m_a\cdot\sqrt{\frac{w_a}{g_a}}\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a n_a \overset{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324} + \frac{(b-c)^2}{9}\cdot\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$

$$2s(s-a)\cdot\frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a)\cdot\frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9}\right) +$$

$$\left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\geq} 0 (\because (b-c)^2 \geq 0)$$

$$(\because (b-c)^2 \geq 0) \Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} +$$

$$\frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4}\cdot(b-c)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3\right)}{36a(2s+a)^2} +$$

$$\frac{(s-a)\left((s-a)(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3) + 1701a^4\right)}{324a(2s+a)^4}\cdot(b-c)^2$$

$$\overset{?}{\geq} 0 \rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \rightarrow (1)$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\geq} m_a w_a \to (2) \therefore (1)\bullet(2) \Rightarrow (m_a n_a)(n_a g_a) \geq p_a^2 . m_a w_a$$

$$\Rightarrow n_a \geq \sqrt{\frac{w_a}{g_a}}.p_a \text{ and } (2) \Rightarrow n_a \geq \sqrt{\frac{w_a}{g_a}}.m_a.\sqrt{\frac{w_a}{g_a}}$$

$$\therefore 2n_a \geq \sqrt{\frac{w_a}{g_a}}.p_a + \sqrt{\frac{w_a}{g_a}}.m_a.\sqrt{\frac{w_a}{g_a}} \text{ and so, } 2n_a \geq \sqrt{\frac{w_a}{g_a}}.\left(p_a + m_a.\sqrt{\frac{w_a}{g_a}}\right)$$

$$\forall \Delta ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$

3727. In any Δ ABC the following relationship holds :

$$\left(\frac{n_a + n_b + n_c}{s}\right)^2 + 2\left(\frac{h_a}{r_b} + \frac{h_b}{r_c} + \frac{h_c}{r_a}\right) \leq 1 + \frac{4R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} n_a = \sum_{cyc}\left(\frac{n_a}{\sqrt{h_a}}.\sqrt{h_a}\right) \overset{CBS}{\leq} \sqrt{\sum_{cyc}\frac{n_a^2}{h_a}}.\sqrt{\sum_{cyc} h_a} \overset{\text{Bogdan Fustei}}{=}$$

$$\sqrt{s^2\sum_{cyc}\frac{1}{h_a} - 2\sum_{cyc} r_a}.\sqrt{\frac{s^2 + 4Rr + r^2}{2R}}$$

$$\therefore \left(\frac{n_a + n_b + n_c}{s}\right)^2 \leq \frac{(s^2 - 8Rr - 2r^2)(s^2 + 4Rr + r^2)}{2Rrs^2} \to (1) \text{ and now, for all}$$

$$x, y, z > 0, \sum_{cyc}\frac{x}{y} = \sum_{cyc}\frac{x^2}{xy} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{cyc} x\right)^2}{\sum_{cyc} xy} \text{ and putting } x \equiv s - a, y \equiv s - b,$$

$$z \equiv s - c \text{ in } \sum_{cyc}\frac{x}{y} \geq \frac{\left(\sum_{cyc} x\right)^2}{\sum_{cyc} xy}, \text{ we get : } \sum_{cyc}\frac{(s-a)(s-b)^2}{(s-a)(s-b)(s-c)} \geq \frac{s^2}{4Rr + r^2}$$

$$\Rightarrow \sum_{cyc}\left((s-a)(s^2 - 2sb + b^2)\right) \geq \frac{rs^3}{4R + r}$$

$$\Rightarrow s^3 - 2s^2(2s) + 2s\sum_{cyc} ab + s\sum_{cyc} a^2 - \sum_{cyc} ab^2 \geq \frac{rs^3}{4R + r}$$

$$\Rightarrow s^3 - 4s^3 + s.4s^2 - \sum_{cyc} ab^2 \geq \frac{rs^3}{4R + r} \Rightarrow \sum_{cyc} ab^2 \leq \frac{4Rs^3}{4R + r} \to (2) \text{ and now,}$$

$$\left(\frac{n_a+n_b+n_c}{s}\right)^2+2\left(\frac{h_a}{r_b}+\frac{h_b}{r_c}+\frac{h_c}{r_a}\right)=\left(\frac{n_a+n_b+n_c}{s}\right)^2+\sum_{cyc}\left(\frac{4rs}{a}\cdot\frac{s-b}{rs}\right)$$

$$=\left(\frac{n_a+n_b+n_c}{s}\right)^2+\frac{1}{Rrs}\cdot\sum_{cyc}\left(bc(s-b)\right)=$$

$$\left(\frac{n_a+n_b+n_c}{s}\right)^2+\frac{1}{Rrs}\cdot\left(s\sum_{cyc}ab-\left(2s\sum_{cyc}ab-12Rrs\right)+\sum_{cyc}ab^2\right)\overset{\text{via ① and ②}}{\leq}$$

$$\frac{(s^2-8Rr-2r^2)(s^2+4Rr+r^2)}{2Rrs^2}-\frac{s^2+4Rr+r^2}{Rr}+12+\frac{4s^2}{r(4R+r)}\overset{?}{\leq}1+\frac{4R}{r}$$

$$\Leftrightarrow(32R^3-32R^2r+2Rr^2+3r^3)s^2+2r^2(4R+r)^3\boxed{\underset{(*)}{\overset{?}{\geq}}}(4R-r)s^4$$

$$\text{Now, }(4R-r)s^4\overset{\text{Gerretsen}}{\leq}(4R-r)(4R^2+4Rr+3r^2)s^2\overset{?}{\leq}\text{RHS of }(*)$$

$$\Leftrightarrow(8R^3-22R^2r-3Rr^2+3r^3)s^2+r^2(4R+r)^3\boxed{\underset{(**)}{\overset{?}{\geq}}}0\text{ and it' strivially true}$$

$$\text{if : }8R^3-22R^2r-3Rr^2+3r^3\geq0\text{ and when : }8R^3-22R^2r-3Rr^2+3r^3<0,$$

$$\text{then : LHS of }(**)\overset{\text{Gerretsen}}{\geq}(8R^3-22R^2r-3Rr^2+3r^3)(4R^2+4Rr+3r^2)+$$
$$r^2(4R+r)^3\overset{?}{\geq}0\Leftrightarrow32t^5-56t^4-12t^3-18t^2+15t+10\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(32t^4+8t^3+4t^2-10t-5)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)$$
$$\Rightarrow(*)\text{ is true}$$

$$\therefore\left(\frac{n_a+n_b+n_c}{s}\right)^2+2\left(\frac{h_a}{r_b}+\frac{h_b}{r_c}+\frac{h_c}{r_a}\right)\leq1+\frac{4R}{r}\ \forall\,\Delta\,ABC,$$
$$''='' \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

3728. In any $\Delta\,ABC$ the following relationship holds :

$$\prod_{cyc}(w_a^2+w_aw_b+w_b^2)\geq\left(\frac{2r}{R}\right)^2\left(\frac{9r_ar_br_c}{r_a+r_b+r_c}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}(w_a^2+w_aw_b+w_b^2)\overset{AM-GM}{\geq}\prod_{cyc}(3w_aw_b)=27.\frac{256R^2r^4s^4}{(s^2+2Rr+r^2)^2}$$

$$\overset{?}{\geq}\left(\frac{2r}{R}\right)^2\left(\frac{9r_ar_br_c}{r_a+r_b+r_c}\right)^3\Leftrightarrow 64R^4(4R+r)^3\overset{?}{\geq}27s^2r(s^2+2Rr+r^2)^2\ and$$

$$\because (4R+r)^3\overset{\substack{Euler\\and\\Doucet\ or\ Trucht}}{\geq}9r.3s^2\ \therefore\ \text{it suffices to prove}:64R^4\overset{?}{\geq}(s^2+2Rr+r^2)^2$$

$$\Leftrightarrow s^2\overset{?}{\leq}8R^2-2Rr-r^2\Leftrightarrow s^2-(4R^2+4Rr+3r^2)-2(R-2r)(2R+r)\overset{?}{\leq}0$$

$$\to \text{true}\because s^2-(4R^2+4Rr+3r^2)\overset{Gerretsen}{\leq}0\ and-2(R-2r)(2R+r)\overset{Euler}{\leq}0$$

$$\therefore\prod_{cyc}(w_a^2+w_aw_b+w_b^2)\geq\left(\frac{2r}{R}\right)^2\left(\frac{9r_ar_br_c}{r_a+r_b+r_c}\right)^3\quad\forall\Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3729. In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{h_a^3-9r^3}{h_a^2}\geq 6r$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a^3-9r^3}{h_a^2}=\sum_{cyc}h_a-\frac{9r^3}{4r^2s^2}.\sum_{cyc}a^2=\frac{s^2+4Rr+r^2}{2R}-\frac{9r(s^2-4Rr-r^2)}{2s^2}$$

$$=\frac{s^2(s^2+4Rr+r^2)-9Rr(s^2-4Rr-r^2)}{2Rs^2}\overset{?}{\geq}6r$$

$$\Leftrightarrow s^4-(17Rr-r^2)s^2+9Rr^2(4R+r)\underbrace{\overset{?}{\geq}}_{(*)}0\ and\because(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\geq}0$$

$$\therefore\ \text{in order to prove }(*),\text{it suffices to prove : LHS of }(*)\overset{?}{\geq}(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow(15R-9r)s^2\underbrace{\overset{?}{\geq}}_{(**)}r(220R^2-169Rr+25r^2)\ and\ indeed,$$

$$(15R-9r)s^2-r(220R^2-169Rr+25r^2)\overset{Gerretsen}{\geq}$$
$$(15R-9r)(16Rr-5r^2)-r(220R^2-169Rr+25r^2)=10r(R-2r)(2R-r)$$

$$\overset{Euler}{\geq}0\Rightarrow(**)\Rightarrow(*)\text{ is true}\therefore\sum_{cyc}\frac{h_a^3-9r^3}{h_a^2}\geq 6r\ \forall\Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3730. In any acute $\Delta\ ABC$ the following relationship holds :

$$\sum_{cyc}\frac{r_a^3-9r^3}{r_a^2}\geq 6r$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_a^3-9r^3}{r_a^2}=\sum_{cyc}r_a-\frac{9r^3}{r^2s^2}\cdot\sum_{cyc}(s-a)^2$$

$$=4R+r-\frac{9r}{s^2}\cdot\left(3s^2-2s(2s)+2(s^2-4Rr-r^2)\right)$$

$$=\frac{(4R+r)s^2-9r(s^2-8Rr-2r^2)}{s^2}\overset{?}{\geq}6r\Leftrightarrow(2R-7r)s^2+9r^2(4R+r)\underset{(*)}{\overset{?}{\geq}}0$$

and it's trivially true when : $2R-7r\geq 0$ and when : $2R-7r<0$, then :

$$\text{LHS of }(*)\overset{\text{Gerretsen}}{\geq}(2R-7r)(4R^2+4Rr+3r^2)+9r^2(4R+r)$$

$$=2(R-2r)(4R^2-2Rr+3r^2)\geq 0\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\frac{r_a^3-9r^3}{r_a^2}\geq 6r\ \forall\ \Delta\ ABC,''=''\ \text{iff }\Delta\ ABC\text{ is equilateral (QED)}$$

3731. In ΔABC the following relationship holds:

$$\sum\left(\frac{r_a}{r_b+r_c}\right)^n\geq\frac{3}{6^n}\left(\frac{4R+r}{s}\right)^{2n},n\in N$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\textit{Known result }\sum\frac{1}{s-a}=\frac{4R+r}{sr}$$

$$\frac{r_a}{r_b+r_c}=\frac{\frac{F}{s\text{-}a}}{\frac{F}{s-b}+\frac{F}{s-c}}=\frac{(s-b)(s-c)}{a(s-a)}=(s-a)(s-b)(s-c)\frac{\left(\frac{1}{s-a}\right)^2}{a}$$

$$\sum\frac{r_a}{r_b+r_c}=(s-a)(s-b)(s-c)\sum\frac{\left(\frac{1}{s-a}\right)^2}{a}\overset{Bergstrom}{\geq}$$

$$\geq (s-a)(s-b)(s-c)\frac{\left(\sum \frac{1}{s-a}\right)^2}{a+b+c}=$$

$$=\frac{(s-a)(s-b)(s-c)\left(\frac{r(4R+r)}{(s-a)(s-b)(s-c)}\right)^2}{a+b+c}=\frac{\left(r(4R+r)\right)^2}{2s.sr^2}=\frac{1}{2}\left(\frac{4R+r}{s}\right)^2$$

$$\sum\left(\frac{r_a}{r_b+r_c}\right)^n \overset{CBS}{\geq} \frac{1}{3^{n-1}}\left(\sum\frac{r_a}{r_b+r_c}\right)^n \geq \frac{1}{3^{n-1}}\left(\frac{1}{2}\left(\frac{4R+r}{s}\right)^2\right)^n=\frac{3}{6^n}\left(\frac{4R+r}{s}\right)^{2n}$$

Equality holds for an equilateral triangle.

3732. In ΔABC the following relationship holds:

$$\sum\left(\frac{r_a}{r_br_c}\right)^3 \geq \frac{1}{9r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ x=\frac{r}{r_a}, y=\frac{r}{r_b}, z=\frac{r}{r_c}\ then\ x+y+z=1\ as$$

$$\sum\frac{1}{r_a}=\frac{1}{r},\qquad \frac{r_a}{r_br_c}=\frac{\frac{1}{r_br_c}}{\frac{1}{r_a}}=\frac{1}{r}\frac{\frac{r}{r_b}\cdot\frac{r}{r_c}}{\frac{r}{r_a}}=\frac{1}{r}\frac{yz}{x}$$

$$\sum\left(\frac{r_a}{r_br_c}\right)^3=\frac{1}{r^3}\sum\left(\frac{yz}{x}\right)^3=\frac{1}{r^3}\frac{y^6z^6+z^6x^6+x^6y^6}{x^3y^3z^3}\overset{\forall a,b,c>0\ \sum a^2\geq\sum ab}{\geq}$$

$$\geq\frac{1}{r^3}\frac{x^3y^3z^3(x^3+y^3+z^3)}{x^3y^3z^3}\overset{CBS}{\geq}\frac{1}{r^3}\frac{1}{9}(x+y+z)^3=\frac{1}{9r^3}$$

Equality holds for an equilateral triangle.

3733. If $n\geq 4$ then in ΔABC the following relationship holds:

$$\frac{m_a}{m_b+m_c}+\frac{m_b}{m_a+m_c}+\frac{m_c}{m_b+m_a}+\left(\frac{R}{2r}\right)^n\geq 1+\frac{w_a}{w_b+w_c}+\frac{w_b}{w_a+w_c}+\frac{w_c}{w_b+w_a}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\frac{m_a}{m_b+m_c}+\frac{m_b}{m_a+m_c}+\frac{m_c}{m_b+m_a}\overset{Nesbitt}{\geq}\frac{3}{2}$$

$$\frac{w_a}{w_b}+\frac{w_b}{w_c}+\frac{w_c}{w_a}\overset{CBS}{\leq}\sqrt{\left(\sum w_a^2\right)\left(\sum\frac{1}{w_a^2}\right)}\leq\sqrt{\left(\sum s(s-a)\right)\left(\sum\frac{1}{h_a^2}\right)}=$$

$$=\sqrt{(s^2)\left(\sum\frac{a^2}{4F^2}\right)}\overset{Leibniz}{\leq}\sqrt{\frac{9R^2}{4r^2}}=\frac{3R}{2r}$$

$$\frac{w_a}{w_b+w_c}+\frac{w_b}{w_a+w_c}+\frac{w_c}{w_b+w_a}=\sum\frac{w_a}{w_b+w_c}\overset{AM-HM}{\leq}\frac{1}{4}\sum\left(\frac{w_a}{w_b}+\frac{w_a}{w_c}\right)=$$

$$=\frac{1}{4}\left(\sum\frac{w_a}{w_b}+\sum\frac{w_b}{w_a}\right)\leq\frac{1}{4}.2\frac{3R}{2r}=\frac{3R}{4r}$$

$$\textit{We need to show}:$$

$$\frac{3}{2}+\left(\frac{R}{2r}\right)^n\leq 1+\frac{3R}{4r}\ or, 2x^n-3x+1\overset{\frac{R}{2r}=x\geq 1\ Euler}{\geq}0$$

$$or\ (x-1)(2x^{n-1}+2x^{n-2}+\cdots 2x^2+2x-1)\geq 0$$

$$true\ as\ x\geq 1\ and\ (2x^{n-1}+2x^{n-2}+\cdots 2x^2+2x-1)>0\ since\ n\geq 4$$

Equality holds for an equilateral triangle.

3734. In ΔABC the following relationship holds:

$$\sqrt{3\sum\frac{a}{b}+\frac{R^2}{r^2}}\geq 4+\max\left\{\sum\frac{a}{b},\sum\frac{b}{a}\right\}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a}{b}\overset{CBS}{\leq}\sqrt{\left(\left(\sum a^2\right)\left(\sum\frac{1}{b^2}\right)\right)}\overset{Leibniz\ \&\ Steinig}{\leq}\sqrt{\frac{9R^2}{4r^2}}=\frac{3R}{2r}$$

$$Similary:\sum\frac{b}{a}\overset{CBS}{\leq}\frac{3R}{2r}$$

$$\max\left\{\sum\frac{a}{b},\sum\frac{b}{a}\right\}\le\frac{3R}{2r},\qquad \sqrt{3\sum\frac{a}{b}}\overset{AM-GM}{\ge}\sqrt{3\times 3}=3$$

$$We\ need\ to\ show:$$

$$\sqrt{3\sum\frac{a}{b}}+\frac{R^2}{r^2}\ge 4+\max\left\{\sum\frac{a}{b},\sum\frac{b}{a}\right\}$$

$$or\ 3+\frac{R^2}{r^2}\ge 4+\frac{3R}{2r}\ or\ 2x^2-3x-2\overset{\frac{R}{r}=x\ge 2\ Euler}{\ge}0\ or\ (x-2)(2x+1)\ge 0$$

$$true\ as\ x\ge 2$$

Equality holds for an equilateral triangle.

3735. If $n\ge 3$ then in ΔABC the following relationship holds:

$$\min\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}+\left(\frac{R}{2r}\right)^n\ge 1+\max\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\sum\frac{a}{a+b}\overset{AM-HM}{\le}\frac{1}{4}\sum\left(1+\frac{a}{b}\right)=\frac{3}{4}+\frac{1}{4}\sum\frac{a}{b}\overset{CBS}{\le}$$

$$\le\frac{3}{4}+\frac{1}{4}\sqrt{\left(\sum a^2\right)\left(\sum\frac{1}{a^2}\right)}\overset{Leibniz\ \&\ Steining}{\le}\frac{3}{4}+\frac{1}{4}\cdot\frac{3R}{2r}=\frac{3}{4}+\frac{3R}{8r}$$

$$Similarly:\sum\frac{b}{a+b}\overset{AM-HM}{\le}\frac{1}{4}\sum\left(1+\frac{b}{a}\right)=\frac{3}{4}-\frac{1}{4}\sum\frac{a}{b}\le\frac{3}{4}+\frac{3R}{8r}$$

$$\max\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}\le\frac{3}{4}+\frac{3R}{8r}$$

$$\sum\frac{a}{a+b}=\sum\left(1-\frac{b}{a+b}\right)=3-\sum\frac{b}{a+b}$$

$$Similarly:\sum\frac{b}{a+b}=\sum\left(1-\frac{a}{a+b}\right)=3-\sum\frac{a}{a+b}$$

$$Then:\min\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}\ge 3-\max\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}$$

$$We\ need\ to\ show:$$

$$\min\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}+\left(\frac{R}{2r}\right)^n\geq 1+\max\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}$$

$$or\ 3-\max\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}+\left(\frac{R}{2r}\right)^n\geq 1+\max\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}$$

$$or\ 2+\left(\frac{R}{2r}\right)^n\geq 2.\max\left\{\sum\frac{a}{a+b},\sum\frac{b}{a+b}\right\}$$

$$or\ 2+\left(\frac{R}{2r}\right)^n\geq 2.\left(\frac{3}{4}+\frac{3R}{8r}\right)or, 2x^n-3x+1\overset{\frac{R}{2r}=x\geq 1\ Euler}{\geq}0$$
$$or(x-1)(2x^{n-1}+2x^{n-2}+\cdots+2x-1)\geq 0$$

$$true\ as\ x\geq 1\ and\ (2x^{n-1}+2x^{n-2}+\cdots+2x-1)>0\ as\ n\geq 3$$

Equality holds for an equilateral triangle.

3736. In ΔABC the following relationship holds:

$$\frac{AH}{a\cdot r_a}+\frac{BH}{b\cdot r_b}+\frac{CH}{c\cdot r_c}\geq\frac{s}{9r^2}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$Lemma\ 1:\ AH=2R.\cos A$
$Lemma\ 2:\ r_a=s\cdot\tan\frac{A}{2},\ \tan\frac{A}{2}=\frac{r}{s-a}$
$Lemma\ 3:\ Gerretsen\ inequality:\ s^2\geq 16rR-5r^2$

$$LHS=\sum\frac{AH}{a\cdot r_a}=\sum\frac{2R\cdot\cos A}{2R\cdot\sin A\cdot r_a}=\sum\frac{\cot A}{r_a}=\sum\frac{\cot A}{s\cdot\tan\frac{A}{2}}=\frac{1}{2s}\sum\frac{1-\tan^2\frac{A}{2}}{\tan^2\frac{A}{2}}=$$

$$=\frac{1}{2s}\left(\left(\sum\frac{1}{\tan^2\frac{A}{2}}\right)-3\right)=\frac{1}{2s}\left(\left(\sum\frac{(s-a)^2}{r^2}\right)-3\right)=$$

$$=\frac{1}{2s}\left(\frac{3s^2-2s(a+b+c)+a^2+b^2+c^2}{r^2}-3\right)=$$

$$=\frac{1}{2s}\left(\frac{-s^2+2s^2-8Rr-2r^2}{r^2}-3\right)=\frac{1}{2s}\cdot\frac{s^2-8Rr-5r^2}{r^2}$$

$$\frac{1}{2s}\cdot\frac{s^2-8Rr-5r^2}{r^2}\geq?\frac{s}{9r^2}\Rightarrow 9s^2-72Rr-45r^2\geq?2s^2\Rightarrow$$

$$7s^2\geq?72Rr+45r^2 \text{ (to prove)}$$

$$7s^2\geq 7(16rR-5r^2)=112Rr-35r^2=$$

$$=72Rr+40Rr-35r^2\geq 72Rr+80r^2-35r^2=72Rr+45r^2\ (true)$$

$$Equality\ holds\ for:\ a=b=c.$$

3737. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{m_a}{h_a\sin\frac{A}{2}}\geq\frac{3R}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a}{h_a\sin\frac{A}{2}}\overset{\text{Lascu}}{\geq}\sum_{cyc}\frac{a(b+c)\cos\frac{A}{2}}{4rs\sin\frac{A}{2}}=\sum_{cyc}\frac{sa(2s-a)(s-a)}{4r^2s^2}=$$
$$=\frac{1}{4r^2s}\cdot\sum_{cyc}\left(a(2s^2-3sa+a^2)\right)=$$
$$=\frac{1}{4r^2s}\cdot\left(2s^2(2s)-6s(s^2-4Rr-r^2)+2s(s^2-6Rr-3r^2)\right)=\frac{12Rrs}{4r^2s}=\frac{3R}{r}$$

and so, $\sum_{cyc}\frac{m_a}{h_a\sin\frac{A}{2}}\geq\frac{3R}{r}\ \forall\ \Delta\ ABC,''=''$ iff Δ ABC is equilateral (QED)

3738. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{h_a+h_b}{r_c}=6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{h_a+h_b}{r_c}=\sum_{cyc}\frac{\frac{2F}{a}+\frac{2F}{b}}{\frac{F}{s-c}}=2\sum_{cyc}\frac{(a+b)(s-c)}{ab}=$$

$$=\sum_{cyc}\frac{(a+b)(a+b-c)}{ab}=\sum_{cyc}\frac{a^2+2ab+b^2-ac-bc}{ab}=$$

$$=\sum_{cyc}\left(2+\frac{a}{b}+\frac{b}{a}-\frac{c}{b}-\frac{c}{a}\right)=$$

$$=6+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-\frac{c}{b}-\frac{a}{c}-\frac{b}{a}-\frac{c}{a}-\frac{a}{b}-\frac{b}{c}=6$$

3739. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin(A)+\sin(C)}{\big(\sin(A)+\sin(B)\big)(\sin(B)+\sin(C))}\geq\sqrt{3}\left(\frac{2r}{R}\right)^3$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{\sin(A)+\sin(C)}{\big(\sin(A)+\sin(B)\big)(\sin(B)+\sin(C))}=$$

$$=\sum_{cyc}\frac{2R(a+c)}{(a+b)(b+c)}=2R.\sum_{cyc}\frac{a+c}{(a+b)(b+c)}=$$

$$=2R\frac{\sum_{cyc}(a+c)^2}{(a+b)(b+c)(a+c)}=4R\frac{\sum_{cyc}a^2+\sum_{cyc}ab}{(a+b)(b+c)(a+c)}=$$

$$=4R.\frac{2(p^2-4Rr-r^2)+p^2+4Rr+r^2}{2p(p^2+2Rr+r^2)}=2R.\frac{3p^2-4Rr-r^2}{p(p^2+2Rr+r^2)}\overset{\substack{Gerretsen\\ Mitrinovic}}{\underset{\sim}{\geq}}$$

$$\geq 2R.\frac{3(12Rr+3r^2)-4Rr-r^2}{\frac{3\sqrt{3}}{2}R(4R^2+4Rr+3r^2+2Rr+r^2)}=2\frac{32Rr+8r^2}{3\sqrt{3}(2R^2+3Rr+2r^2)}\overset{Euler}{\underset{\sim}{\geq}}$$

$$\geq\frac{2(64r^2+8r^2)}{3\sqrt{3}\left(2R^2+\frac{3}{2}R^2+\frac{1}{2}R^2\right)}=\frac{144r^2}{3\sqrt{3}\,.\,4R^2}=\sqrt{3}\,.\,\frac{4r^2}{R^2}\overset{Euler}{\underset{\sim}{\geq}}\sqrt{3}\left(\frac{2r}{R}\right)^3$$

$$Equality\ \ holds\ \ for:\ \ A=B=C.$$

3740. In ΔABC the following relationship holds:

$$\sum\frac{h_b^2}{h_a}\ge 9r$$

Proposed by Marin Chirciu-Romania

Solution by Sarkhan Adgozalov-Georgia

$$\sum\frac{h_b^2}{h_a}\ge\frac{(\sum h_a)^2}{\sum h_a}=\sum h_a=2F\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 2rs\cdot\frac{9}{2s}=9r$$

$$Equality\ holds\ for:\ a=b=c.$$

3741. In ΔABC the following relationship holds:

$$m_a^{h_a}m_b^{h_b}m_c^{h_c}\ge(3r)^{9r}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum h_a\overset{AM\ge HM}{\ge}\frac{9}{\sum\frac{1}{h_a}}=\frac{9}{\frac{1}{r}}=9r$$

$$\sum\frac{h_a}{m_a}\overset{Chebshev}{\le}\frac{1}{3}\left(\sum h_a\right)\left(\sum\frac{1}{m_a}\right)$$

$$\sum\frac{1}{m_a}\le\sum\frac{1}{h_a}=\frac{1}{r}$$

$$m_a^{h_a}m_b^{h_b}m_c^{h_c}\overset{GM-HM}{\ge}\left(\frac{(\sum h_a)}{\sum\frac{h_a}{m_a}}\right)^{\sum h_a}\ge\left(\frac{(\sum h_a)}{\frac{1}{3}(\sum h_a)\left(\sum\frac{1}{m_a}\right)}\right)^{\sum h_a}\ge\left(\frac{3}{\frac{1}{r}}\right)^{9r}=(3r)^{9r}$$

Equality holds for an equilateral triangle.

3742. In any Δ ABC the following relationship holds :

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}\le 2\cos\frac{A-B}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}=2\sin\frac{3A+3B}{4}\cos\frac{3A-3B}{4}\le 2\cos\frac{3A-3B}{4}$$

$$\left(\because 0<\sin\frac{3A+3B}{4}\le 1\ as\ 0<\frac{3A+3B}{4}<\frac{3\pi}{4}\right)\overset{?}{\le} 2\cos\frac{A-B}{2}=2\cos^2\frac{A-B}{4}-1$$

$$\Leftrightarrow 4\cos^3\theta-3\cos\theta\overset{?}{\le}2\cos^2\theta-1\ \left(\theta=\frac{A-B}{4}\right)$$

$$\Leftrightarrow (t-1)(2t^2-1+2t^2+2t)\overset{?}{\le}0\ (t=\cos\theta)\to \text{true}\because -\frac{\pi}{4}<\theta=\frac{A-B}{4}<\frac{\pi}{4}$$

$$\Rightarrow\frac{1}{\sqrt{2}}<t\le 1\Rightarrow 2t^2-1>0\ \therefore \sin\frac{3A}{2}+\sin\frac{3B}{2}\le 2\cos\frac{A-B}{2}\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{ iff }\ \cos\frac{A-B}{4}=1\ \wedge\ \sin\frac{3A+3B}{4}=1\Rightarrow \text{iff } A=B \text{ and subsequently iff}$$

$$\sin\frac{3A}{2}=1 \text{ and} \because 0<\frac{3A}{2}<\frac{3\pi}{2}\ \therefore \sin\frac{3A}{2}=1 \text{ iff } \frac{3A}{2}=\frac{\pi}{2}\Rightarrow$$

$$\text{iff } A=\frac{\pi}{3} \text{ and so, in conclusion, }''=''\ \text{ iff } A=B=\frac{\pi}{3}\ (QED)$$

3743. In any acute Δ ABC the following relationship holds :

$$\frac{\sqrt{\sin^2A+\sin^2B}}{\cos C}+\frac{\sqrt{\sin^2B+\sin^2C}}{\cos A}+\frac{\sqrt{\sin^2C+\sin^2A}}{\cos B}\ge\sqrt{54}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\sqrt{\sin^2B+\sin^2C}}{\cos A}\ge\frac{1}{\sqrt{2}}.\sum_{cyc}\frac{\sin B+\sin C}{\cos A}$$

$$=\frac{1}{\sqrt{2}}.\sum_{cyc}\left((\sin A)\left(\frac{1}{\cos B}+\frac{1}{\cos C}\right)\right)\overset{\text{Bergstrom}}{\ge}\frac{1}{\sqrt{2}}.\sum_{cyc}\left((\sin A)\left(\frac{4}{\cos B+\cos C}\right)\right)$$

$$=2\sqrt{2}.\sum_{cyc}\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{B-C}{2}}=4\sqrt{2}.\sum_{cyc}\frac{\cos^2\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}=4\sqrt{2}.\sum_{cyc}\frac{\cos^2\frac{A}{2}}{\sin B+\sin C}$$

$$\overset{\text{Bergstrom}}{\ge}4\sqrt{2}.\frac{\left(\sum_{cyc}\cos\frac{A}{2}\right)^2}{2\sum_{cyc}\sin A}=2\sqrt{2}.\frac{R}{s}.\left(\sum_{cyc}\cos^2\frac{A}{2}+2\sum_{cyc}\left(\cos\frac{B}{2}\cos\frac{C}{2}\right)\right)$$

$$=2\sqrt{2}.\frac{R}{s}.\left(\frac{4R+r}{2R}+\frac{2s}{4R}.\sum_{cyc}\sec\frac{A}{2}\right)\overset{Jensen}{\geq}\frac{\sqrt{2}}{s}.\left(4R+r+s.3.\frac{2}{\sqrt{3}}\right)\overset{Doucet\ or\ Trucht}{\geq}$$

$$\frac{\sqrt{2}}{s}.\left(s.\sqrt{3}+2s.\sqrt{3}\right)=3.\sqrt{6}=\sqrt{54}\ and\ so,\sum_{cyc}\frac{\sqrt{\sin^2B+\sin^2C}}{\cos A}\geq\sqrt{54}$$

$\forall$ *acute* Δ ABC, " = " iff Δ ABC is equilateral (QED)

3744. In ΔABC the following relationship holds:

$$\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}}+\frac{\cos\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}}\geq 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{\cos\frac{A}{2}}{\sin\frac{B}{2}}\frac{\cos\frac{B}{2}}{\sin\frac{C}{2}}\cdot\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}}}=3\sqrt[3]{\frac{s\sqrt{s(s-a)(s-b)(s-c)}}{(s-a)(s-b)(s-c)}}=$$

$$3\sqrt[3]{\frac{s^2F}{F^2}}=3\sqrt[3]{\frac{s^2}{rs}}=3\sqrt[3]{\frac{s}{r}}\overset{MITRINOVICI}{\geq}3\sqrt[3]{\frac{3\sqrt{3}r}{r}}=3\sqrt[3]{\left(\sqrt{3}\right)^3}=3\sqrt{3}$$

Equality holds for $A=B=C$.

3745. In any Δ ABC the following relationship holds :

$$\frac{(n_b+n_c)^2}{\sum_{cyc}h_ah_b}>\frac{R}{2r}\geq\frac{5R-r+\sum_{cyc}\frac{n_ag_a}{h_a}}{2\sum_{cyc}h_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{n_ag_a}{h_a}\overset{AM-GM}{\leq}\sum_{cyc}\frac{a(n_a^2+g_a^2)}{4rs}\overset{Bogdan\ Fustei}{=}\sum_{cyc}\frac{a\left(4m_a^2-2s(s-a)\right)}{4rs}$$

$$=\frac{1}{4rs}\cdot\left(\sum_{cyc}\left(a\left(2\sum_{cyc}a^2-3a^2\right)\right)-2s\left(s(2s)-2(s^2-4Rr-r^2)\right)\right)$$

$$=\frac{8s(s^2-4Rr-r^2)-6s(s^2-6Rr-3r^2)-4s(4Rr+r^2)}{4rs}=\frac{s^2-6Rr+3r^2}{2r}$$

$$\therefore 5R-r+\sum_{cyc}\frac{n_a g_a}{h_a}\le 5R-r+\frac{s^2-6Rr+3r^2}{2r}=\frac{s^2+4Rr+r^2}{2r}=\frac{R}{r}\cdot\sum_{cyc}h_a$$

$$\Rightarrow\frac{R}{2r}\ge\frac{5R-r+\sum_{cyc}\frac{n_a g_a}{h_a}}{2\sum_{cyc}h_a}\ \text{and again},\frac{(n_b+n_c)^2}{\sum_{cyc}h_a h_b}>\frac{s^2}{\sum_{cyc}h_a h_b}$$

(Reference → "Nagel's Cevians Revisited (ii)" by Bogdan Fustei; published at : www.ssmrmh.ro)

$$=\frac{s^2}{\frac{4Rrs(2s)}{4R^2}}=\frac{R}{2r}\ \text{and so},\frac{(n_b+n_c)^2}{\sum_{cyc}h_a h_b}>\frac{R}{2r}\ge\frac{5R-r+\sum_{cyc}\frac{n_a g_a}{h_a}}{2\sum_{cyc}h_a}\ \forall\ \Delta\ ABC,$$

" =" iff Δ ABC is equilateral (QED)

3756. In ΔABC the following relationship holds:

$$\left(3+\sum_{cyc}\frac{a^2+b^2}{c^2}\right)\left(\sum_{cyc}rr_a\right)\le\left(\frac{9}{4}\cdot\frac{R^2}{r}\right)^2$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$3+\sum_{cyc}\frac{a^2+b^2}{c^2}=\frac{a^2+b^2+c^2}{a^2}+\frac{a^2+b^2+c^2}{b^2}+\frac{a^2+b^2+c^2}{c^2}=$$

$$=(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\overset{\substack{Leibniz\\ Steining}}{\le}9R^2\cdot\frac{1}{4r^2}=\frac{9R^2}{4r^2}\ (1)$$

$$\sum_{cyc}rr_a=r\sum_{cyc}r_a=r(4R+r)\overset{Euler}{\le}\frac{9R^2}{4}\ (2)$$

From (1) and (2) we have

$$\left(3+\sum_{cyc}\frac{a^2+b^2}{c^2}\right)\left(\sum_{cyc}rr_a\right)\overset{\substack{(1)\\(2)}}{\le}\left(\frac{9}{4}\cdot\frac{R^2}{r}\right)^2$$

Equality holds for : $a=b=c$

3747. In ΔABC the following relationship holds:

$$\frac{1}{s}\left(\frac{4R}{r}-5\right)\le\sum\frac{AH}{ar_a}\le\frac{1}{s}\left(\frac{2R^2}{r^2}-5\right)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Known\ Results: 1)AH=2R\cos A, 2)r_a=s\tan\frac{A}{2}$$

$$3)\sum\tan^2\frac{A}{2}.\tan^2\frac{B}{2}=\frac{s^2-2r^2-8Rr}{s^2}\quad 4)\prod\tan^2\frac{A}{2}=\left(\frac{r}{s}\right)^2$$

$$\sum\cot^2\frac{A}{2}=\frac{\sum\tan^2\frac{A}{2}.\tan^2\frac{B}{2}}{\prod\tan^2\frac{A}{2}}=\frac{s^2-2r^2-8Rr}{r^2}$$

$$\frac{AH}{ar_a}=\frac{2R\cos A}{2RsinA.s\,tan\frac{A}{2}}=\frac{cotA}{s\tan\frac{A}{2}}=\frac{\frac{\cot^2\frac{A}{2}-1}{2\cot\frac{A}{2}}}{s\tan\frac{A}{2}}=\frac{1}{2s}\left(\cot^2\frac{A}{2}-1\right)$$

$$\sum\frac{AH}{ar_a}=\frac{1}{2s}\sum\left(\cot^2\frac{A}{2}-1\right)=\frac{1}{2s}\left(\frac{s^2-2r^2-8Rr}{r^2}-3\right)=$$

$$=\frac{1}{2s}\frac{s^2-5r^2-8Rr}{r^2}\overset{Gerretsen}{\le}\frac{1}{2s}\frac{4R^2+4Rr+3r^2-5r^2-8Rr}{r^2}=$$

$$=\frac{1}{2s}\left(\frac{4R^2}{r^2}-\frac{4R}{r}-2\right)\overset{Euler}{\le}\frac{1}{2s}\left(\frac{4R^2}{r^2}-4\times2-2\right)=\frac{1}{s}\left(\frac{2R^2}{r^2}-5\right)$$

$$\sum\frac{AH}{ar_a}=\frac{1}{2s}\frac{s^2-5r^2-8Rr}{r^2}\overset{Gerretsen}{\ge}\frac{1}{2s}\frac{16Rr-5r^2-5r^2-8Rr}{r^2}=\frac{1}{s}\left(\frac{4R}{r}-5\right)$$

Equality holds for an equilateral triangle.

3748. In Δ ABC with $n_a, n_b, n_c \to$ *Nagel* cevians then :

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\ge\sqrt{\frac{s^2-12Rr}{r^2}+6}\ge\sqrt{\frac{4R}{r}+1}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{n_a^2}{h_a^2}\overset{\text{Bogdan Fustei}}{=}\sum_{cyc}\frac{a^2s^2-\frac{r}{R}.s^2\sec^2\frac{A}{2}.16R^2\cos^2\frac{A}{2}\sin^2\frac{A}{2}}{4r^2s^2}$$

$$=\frac{s^2-4Rr-r^2-8Rr.\frac{2R-r}{2R}}{2r^2}\therefore\sum_{cyc}\frac{n_a^2}{h_a^2}\overset{①}{=}\frac{s^2-12Rr+3r^2}{2r^2}$$

$$\text{Now, }\frac{n_a^2-h_a^2}{h_a^2}=\frac{s(s-a)+\frac{s}{a}(b-c)^2-\left(s(s-a)-\frac{s(s-a)}{a^2}(b-c)^2\right)}{h_a^2}$$

$$=\frac{\frac{s^2}{a^2}}{\frac{4r^2s^2}{a^2}}.(b-c)^2=\frac{(b-c)^2}{4r^2}\therefore\frac{n_a}{h_a}=\sqrt{1+\frac{(b-c)^2}{4r^2}}\text{ and analogs}\therefore 2\sum_{cyc}\left(\frac{n_b}{h_b}.\frac{n_c}{h_c}\right)=$$

$$2\sum_{cyc}\sqrt{\left(1+\frac{(c-a)^2}{4r^2}\right).\left(1+\frac{(a-b)^2}{4r^2}\right)}\overset{\text{Reverse CBS}}{\ge}2\sum_{cyc}\left(1+\frac{|c-a||a-b|}{4r^2}\right)$$

$$=6+\sum_{cyc}\frac{|a-b|.(|b-c|+|c-a|)}{4r^2}\overset{\text{Triangle Inequality}}{\ge}$$

$$6+\sum_{cyc}\frac{|a-b|.(|b-c+c-a|)}{4r^2}=6+\sum_{cyc}\frac{(a-b)^2}{4r^2}=6+\frac{1}{2r^2}.\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)$$

$$\therefore 2\sum_{cyc}\left(\frac{n_b}{h_b}.\frac{n_c}{h_c}\right)\overset{②}{\ge}6+\frac{s^2-12Rr-3r^2}{2r^2}\therefore\sum_{cyc}\frac{n_a^2}{h_a^2}+2\sum_{cyc}\left(\frac{n_b}{h_b}.\frac{n_c}{h_c}\right)\overset{\text{via }①+②}{\ge}$$

$$\frac{s^2-12Rr+3r^2}{2r^2}+6+\frac{s^2-12Rr-3r^2}{2r^2}=\frac{s^2-12Rr}{r^2}+6$$

$$\therefore\left(\sum_{cyc}\frac{n_a}{h_a}\right)^2\ge\frac{s^2-12Rr}{r^2}+6\Rightarrow\sum_{cyc}\frac{n_a}{h_a}\ge\sqrt{\frac{s^2-12Rr}{r^2}+6}\overset{\text{Gerretsen}}{\ge}$$

$$\sqrt{\frac{16Rr-5r^2-12Rr}{r^2}+6}=\sqrt{\frac{4R}{r}+1}\text{ and so,}$$

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\ge\sqrt{\frac{s^2-12Rr}{r^2}+6}\ge\sqrt{\frac{4R}{r}+1}\ \forall\ \Delta\ ABC,$$
$$''=''\ \text{iff}\ \Delta\ ABC\ \text{is equilateral (QED)}$$

3749. In Δ ABC with $n_a, n_b, n_c \to$ Nagel cevians holds :

$$\frac{n_a+n_b+n_c}{h_a+h_b+h_c}\ge\frac{3R}{R+4r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{n_a+n_b+n_c}{h_a+h_b+h_c}\ge\frac{\sqrt{9s^2-80Rr-2r^2}}{s^2+4Rr+r^2}.2R$$
$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba} - 74;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right)\overset{?}{\ge}\frac{3R}{R+4r}$$
$$\overset{\text{squaring}}{\Leftrightarrow}-9s^4+(36R^2+216Rr+558r^2)s^2-$$
$$r(320R^3+2712R^2r+5256Rr^2+137r^3)\underset{(*)}{\overset{?}{\ge}}0$$
$$\text{Now, since } P=-9(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)\overset{\text{Double-Rouche}}{\ge}0$$
$$\therefore \text{ in order to prove } (*), \text{it suffices to prove : LHS of } (*)\overset{?}{\ge}P$$
$$\Leftrightarrow (9R+144r)s^2+64R^3-570R^2r-1287Rr^2-32r^3\underset{(**)}{\overset{?}{\ge}}0$$
$$\text{Again, LHS of } (**)\overset{\text{Gerretsen}}{\ge}(9R+144r)(16Rr-5r^2)+$$
$$64R^3-570R^2r-1287Rr^2-32r^3\overset{?}{\ge}0\Leftrightarrow 32R^3-213R^2r+486Rr^2-376r^3\overset{?}{\ge}0$$
$$\Leftrightarrow (R-2r)(32R^2-149Rr+188r^2)\overset{?}{\ge}0\to\text{true}\because R-2r\overset{\text{Euler}}{\ge}0\ \text{and}$$
$$\text{discriminant of } 32R^2-149Rr+188r^2=149^2.r^2-128.188.r^2=-1863r^2<0$$
$$\Rightarrow 32R^2-149Rr+188r^2>0\Rightarrow(**)\Rightarrow(*)\ \text{is true}\therefore\frac{n_a+n_b+n_c}{h_a+h_b+h_c}\ge\frac{3R}{R+4r}$$
$$\forall\ \Delta\ ABC,''=''\ \text{iff}\ \Delta\ ABC\ \text{is equilateral (QED)}$$

3750. In any Δ ABC the following relationship holds :

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\ge\frac{3}{2}.\sqrt{\frac{R}{r}+2}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{m_a^2-h_a^2}{h_a^2}=\frac{s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)}{a^2}(b-c)^2\right)}{h_a^2}$$

$$=\frac{\frac{4s^2-4sa+a^2}{4a^2}}{\frac{4r^2s^2}{a^2}}.(b-c)^2=\frac{(2s-a)^2(b-c)^2}{16r^2s^2}=\frac{(b+c)^2(b-c)^2}{16r^2s^2}=\frac{(b^2-c^2)^2}{16r^2s^2}$$

$$\therefore \frac{m_a}{h_a}=\sqrt{1+\frac{(b^2-c^2)^2}{16r^2s^2}}\ \textit{and analogs}\ \therefore 2\sum_{cyc}\left(\frac{m_b}{h_b}.\frac{m_c}{h_c}\right)=$$

$$2\sum_{cyc}\sqrt{\left(1+\frac{(c^2-a^2)^2}{16r^2s^2}\right).\left(1+\frac{(a^2-b^2)^2}{16r^2s^2}\right)}\overset{\text{Reverse CBS}}{\geq}$$

$$2\sum_{cyc}\left(1+\frac{|c^2-a^2||a^2-b^2|}{16r^2s^2}\right)=6+\sum_{cyc}\frac{|a^2-b^2|.\left(|b^2-c^2|+|c^2-a^2|\right)}{16r^2s^2}$$

$$\overset{\text{Triangle Inequality}}{\geq}6+\sum_{cyc}\frac{|a^2-b^2|.\left(|b^2-c^2+c^2-a^2|\right)}{16r^2s^2}=6+\sum_{cyc}\frac{(a^2-b^2)^2}{16r^2s^2}$$

$$=6+\frac{2\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)-2\sum_{cyc}a^2b^2}{16r^2s^2}=6+\frac{2\sum_{cyc}a^2b^2-32r^2s^2}{16r^2s^2}\ \textit{and so,}$$

$$\sum_{cyc}\frac{m_a^2}{h_a^2}+2\sum_{cyc}\left(\frac{m_b}{h_b}.\frac{m_c}{h_c}\right)\geq\sum_{cyc}\frac{a^2\left(2\sum_{cyc}a^2-3a^2\right)}{16r^2s^2}+\frac{2\sum_{cyc}a^2b^2+64r^2s^2}{16r^2s^2}$$

$$=\frac{2\left(\sum_{cyc}a^2\right)^2-3\left(2\sum_{cyc}a^2b^2-16r^2s^2\right)+2\sum_{cyc}a^2b^2+64r^2s^2}{16r^2s^2}$$

$$=\frac{2(s^2-4Rr-r^2)^2-((s^2+4Rr+r^2)^2-16Rrs^2)+28r^2s^2}{4r^2s^2}\overset{?}{\geq}\frac{9}{4}.\frac{R+2r}{r}$$

$$\Leftrightarrow s^4-(17Rr-4r^2)s^2+r^2(4R+r)^2\ \boxed{\underset{①}{\overset{?}{\geq}}}\ 0\ \textit{and}\ \because (s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{in order to prove ①, it suffices to prove : LHS of ①}\overset{?}{\geq}(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (5R-2r)s^2\ \boxed{\underset{②}{\overset{?}{\geq}}}\ r(80R^2-56Rr+8r^2)\ \&\ \text{now,}\ (R-r)(s^2-16Rr+5r^2)\overset{\text{Rouche}}{\geq}$$

$$(R-r)\left(2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right)$$

$$=(R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2+r^2\right)\geq r^2(R-2r)\ \left(\because R-2r\overset{\text{Euler}}{\geq}0\right)$$

$$\Rightarrow s^2\geq 16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\ \therefore (5R-2r)s^2\geq$$

$$(5R-2r)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right)\overset{?}{\ge} r(80R^2-56Rr+8r^2)$$

$$\Leftrightarrow r^2(R-2r)(4R-r)\overset{?}{\ge}0 \to \text{true} \because R\overset{Euler}{\ge}2r \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$\therefore \frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\ge\frac{3}{2}\cdot\sqrt{\frac{R}{r}+2}\ \forall\,\Delta\,ABC,''=''\ \text{iff}\ \Delta\,ABC \text{ is equilateral (QED)}$$

3751. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{cosA}{(1+tan^2A)^2}\ge\frac{3}{32}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\cos^2A=3-\sum\sin^2A=3-\frac{\sum a^2}{4R^2}\overset{Leibniz}{\ge}3-\frac{9R^2}{4R^2}=\frac{3}{4}$$

$$\sum_{cyc}\frac{cosA}{(1+tan^2A)^2}=\sum\frac{\cos A}{.(\sec^2A)^2}=\sum\cos^5A=$$

$$=\sum(\cos^2A)^{\frac{5}{2}}\overset{CBS}{\ge}3.\left(\frac{\sum\cos^2A}{3}\right)^{\frac{5}{2}}\ge3.\left(\frac{\frac{3}{4}}{3}\right)^{\frac{5}{2}}=\frac{3}{32}$$

Equality holds for an equilateral triangle.

3752. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{sin\frac{A}{2}}{\left(1+cot^2\frac{A}{2}\right)^2}\ge\frac{3}{32}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum\sin^2\frac{A}{2}=1-\frac{r}{2R}\overset{Euler}{\ge}1-\frac{1}{4}=\frac{3}{4}$$

$$\sum_{cyc} \frac{\sin\frac{A}{2}}{\left(1+\cot^2\frac{A}{2}\right)^2} = \sum \frac{\sin\frac{A}{2}}{\left(\csc^2\frac{A}{2}\right)^2} = \sum \sin^5\frac{A}{2} =$$

$$= \sum \left(\sin^2\frac{A}{2}\right)^{\frac{5}{2}} \overset{CBS}{\geq} 3.\left(\frac{\sum \sin^2\frac{A}{2}}{3}\right)^{\frac{5}{2}} \geq 3.\left(\frac{\frac{3}{4}}{3}\right)^{\frac{5}{2}} = \frac{3}{32}$$

Equality holds for an equilateral triangle.

3753. In acute ΔABC the following relationship holds:

$$(2-\cos^2 A)(2-\cos^2 B)(2-\cos^2 C) > 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$cos2A + cos2B + cos2C = 2\cos(A+B)\cos(A-B) + 2\cos^2 C - 1 =$$

$$\overset{A+B+C=\pi}{=} -2cosCcos(A-B) + 2\cos^2 C - 1 = -1 + 2cosC(cosC - \cos(A-B)) =$$

$$\overset{A+B+C=\pi}{=} 2cosC(-\cos(A+B) - \cos(A-B)) - 1 = -1 - 4cosAcosBcosC$$

$$\sin^2 A + \sin^2 B + \sin^2 C = \sum \sin^2 A =$$

$$= \frac{1}{2}\sum 2\sin^2 A = \frac{1}{2}\sum(1-\cos 2A) = \frac{3-(cos2A+cos2B+cos2C)}{2} =$$
$$= \frac{1}{2}(3+1+4cosAcosBcosC) = 2 + 2cosAcosBcosC >$$

$$> 2 \ (\text{in acute } cosAcosBcosC > 0)$$

$$\text{Let } \sin^2 A = x, \sin^2 B = y, \sin^2 C = z \text{ and } x+y+z > 2$$

$$\text{as } x^2 = \sin^4 A < \sin^2 A = x \text{ as } 0 < \sin A < 1 \text{ for this } x^2+y^2+z^2 < x+y+z \ (1)$$

$$(2-\cos^2 A)(2-\cos^2 B)(2-\cos^2 C) = (1+\sin^2 A)(1+\sin^2 B)(1+\sin^2 C) =$$

$$= (1+x)(1+y)(1+z) = 1 + (x+y+z) + (xy+yz+zx) + xyz >$$

$$> 1+(x+y+z)+(xy+yz+zx)=1+(x+y+z)+\frac{(x+y+z)^2-(x^2+y^2+z^2)}{2}$$

$$\overset{(1)}{>} 1+(x+y+z)+\frac{(x+y+z)^2-(x+y+z)}{2}=\frac{2+(x+y+z)+(x+y+z)^2}{2}>$$

$$>\frac{2+2+2^2}{2}=4 \text{ as } x+y+z>2$$

3754. In any acute Δ ABC the following relationship holds :

$$\frac{\cos A\cos B}{\cos C}+\frac{\cos B\cos C}{\cos A}+\frac{\cos C\cos A}{\cos B}+4\cos A\cos B\cos C\ge 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\sum_{cyc}\frac{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{r}{R}$$

$$=\frac{r^2}{16R^2}\cdot\frac{4R}{r}\cdot\sum_{cyc}\frac{1}{\sin^2\frac{A}{2}}+\frac{r}{R}=\frac{1}{4Rr}\sum_{cyc}AI^2+\frac{r}{R}=\frac{1}{4Rr}\left(\sum_{cyc}bc-12Rr\right)+\frac{r}{R}$$

$$=\frac{s^2-8Rr+r^2+4r^2}{4Rr}\overset{\text{Gerretsen}}{\ge}\frac{8Rr}{4Rr}=2\ \therefore\ \sum_{cyc}\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\sin\frac{A}{2}}+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\ge 2$$

and implementing it on a triangle with angles : $(\pi-2A),(\pi-2B),(\pi-2C)$,

$$\text{we get : }\sum_{cyc}\frac{\sin\frac{\pi-2B}{2}\sin\frac{\pi-2C}{2}}{\sin\frac{\pi-2A}{2}}+4\sin\frac{\pi-2A}{2}\sin\frac{\pi-2B}{2}\sin\frac{\pi-2C}{2}\ge 2$$

$\Rightarrow \sum_{cyc}\frac{\cos B\cos C}{\cos A}+4\cos A\cos B\cos C\ge 2$ and since the latter triangle is an acute one, hence : $\frac{\cos A\cos B}{\cos C}+\frac{\cos B\cos C}{\cos A}+\frac{\cos C\cos A}{\cos B}+4\cos A\cos B\cos C\ge 2$

$\forall$ acute Δ ABC, "=" iff Δ ABC is equilateral (QED)

3755. In acute ΔABC the following relationship holds:

$$a\tan\left(\frac{A}{2}\right)\sqrt{\sin A}+b\tan\left(\frac{B}{2}\right)\sqrt{\sin B}+c\tan\left(\frac{C}{2}\right)\sqrt{\sin C}>4r$$

Proposed by Vasile Mircea Popa-Romania

Solution by Amin Hajiyev-Azerbaijan

$$\sum_{cyc} a\tan\left(\frac{A}{2}\right)\sin(A) > 4r$$

Lemma 1 (The Relationship between Inradius and Tangent):

$$\tan\left(\frac{A}{2}\right) = \frac{r}{s-a}, \tan\left(\frac{B}{2}\right) = \frac{r}{s-b}, \tan\left(\frac{C}{2}\right) = \frac{r}{s-c}$$

Lemma 2 (Inequality of Sine in Acute Triangles):

$$interval\ \left(0;\frac{\pi}{2}\right) \quad 0 < \sin A < 1 \quad \begin{cases}\sqrt{\sin A} > \sin A\\ \sqrt{\sin B} > \sin B\\ \sqrt{\sin C} > \sin C\end{cases}$$

Lemma 3 (Convexity and Jensen's Inequality):

$$f(a) = \frac{a^2}{s-a}\ \ for\ a \in (0;s)$$

Since its second derivative $f''(a) = \frac{2s^2}{(s-a)^3}$ is strictly positive, the function is strictly convex. By Jensen's Inequality, the cyclic sum $\sum_{cyc}\frac{a^2}{s-a}$ is minimized when a = b = c, corresponding to an equilateral triangle.

$$LHS \overset{Lemma\ 1}{\triangleq} \sum_{cyc} a\left(\frac{r}{s-a}\right)\sqrt{\sin A}$$

$$\underline{Lemma\ 2} \rightarrow r\sum_{cyc}\left(\frac{a}{s-a}\right)\sqrt{\sin A} > r\sum_{cyc}\left(\frac{a}{s-a}\right)\sin A$$

$$Theorem\ sine\ \ \sin A = \frac{a}{2R}\ \ \sin B = \frac{b}{2R}\ \ \sin C = \frac{c}{2R}$$

$$r\sum_{cyc}\frac{a}{s-a}\sin A = \frac{r}{2R}\sum_{cyc}\frac{a^2}{s-a}$$

$$Lemma\ 3: \left(\sum_{cyc}\frac{a}{s-a}\right)_{min} \rightarrow a = b = c\ \ s = \frac{a+b+c}{2} = \frac{3a}{2}, s-a = \frac{a}{2}$$

$$\sum_{cyc}\frac{a^2}{s-a} \geq 3\frac{a^2}{s-a} = 6a \rightarrow LHS > \frac{r}{2R}6a = \frac{3ar}{R}$$

$$a = b = c = R\sqrt{3}\ \ LHS > \frac{3ar}{R} = 3r\sqrt{3}$$

$$3\sqrt{3} > 4 \rightarrow \sum_{cyc} a\tan\left(\frac{A}{2}\right)\sin(A) > 4r$$

3756. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^3\frac{C}{2}+\sin^3\frac{A}{2}\right)}{\sin^2\frac{B}{2}\sin\frac{C}{2}\left(\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}\right)}\geq 24$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A', B', C', x', y', z' > 0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\underbrace{\overset{\text{Walter Janous}}{\geq}}_{①}\sqrt{3\sum_{cyc}A'B'}$$

$$\text{Now, }\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^3\frac{C}{2}+\sin^3\frac{A}{2}\right)}{\sin^2\frac{B}{2}\sin\frac{C}{2}\left(\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}\right)}$$

$$=\sum_{cyc}\frac{\sin^2\frac{A}{2}\sin\frac{C}{2}\cdot\left(\dfrac{\sin^3\frac{C}{2}+\sin^3\frac{A}{2}}{\sin\frac{C}{2}\sin\frac{A}{2}}\right)}{\sin^2\frac{B}{2}\sin\frac{C}{2}\left(\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}\right)}$$

$$=\sum_{cyc}\frac{\dfrac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\cdot\left(\dfrac{\sin^3\frac{C}{2}+\sin^3\frac{A}{2}}{\sin^3\frac{C}{2}\sin^3\frac{A}{2}}\right)}{\dfrac{\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}}{\sin^2\frac{C}{2}\sin^2\frac{A}{2}}}=\sum_{cyc}\frac{\dfrac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\cdot\left(\dfrac{1}{\sin^3\frac{C}{2}}+\dfrac{1}{\sin^3\frac{A}{2}}\right)}{\dfrac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\dfrac{\sin^2\frac{C}{2}}{\sin^2\frac{A}{2}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}, y'=\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}, z'=\frac{\sin^2\frac{C}{2}}{\sin^2\frac{A}{2}}, A'=\frac{1}{\sin^3\frac{B}{2}}, B'=\frac{1}{\sin^3\frac{C}{2}}, C'=\frac{1}{\sin^3\frac{A}{2}}\right)$$

$$\overset{\text{via }①}{\geq}\sqrt{3\sum_{cyc}\frac{1}{\sin^3\frac{B}{2}\sin^3\frac{C}{2}}}\overset{AM-GM}{\geq}3.\sqrt[6]{\frac{1}{\sin^6\frac{A}{2}\sin^6\frac{B}{2}\sin^6\frac{C}{2}}}=\frac{12R}{r}\overset{\text{Euler}}{\geq}24$$

$$\therefore\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^3\frac{C}{2}+\sin^3\frac{A}{2}\right)}{\sin^2\frac{B}{2}\sin\frac{C}{2}\left(\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}\right)}\geq 24\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3757. In acute ΔABC the following relationship holds:

$$\frac{tanB}{tanC}=\frac{sin^2B}{sin^2C}\Rightarrow b=c$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{tanB}{tanC}=\frac{sin^2B}{sin^2C}\Rightarrow\frac{sinB}{sinC}\cdot\frac{cosC}{cosB}=\frac{sin^2B}{sin^2C}\Rightarrow\frac{cosC}{cosB}=\frac{sinB}{sinC}$$

$$sinBcosB=sinCcosC\Rightarrow 2sinBcosB=2sinCcosC$$

$$sin2B=sin2C\Rightarrow sin2B-sin2C=0$$

$$2sin(B-C)cos(B+C)=0\Rightarrow sin(B-C)=0$$

$$B-C=0\Rightarrow B=C\Rightarrow b=c$$

3758. In acute ΔABC the following relationship holds:

$$\frac{BC}{HA}+\frac{CA}{HB}+\frac{AB}{HC}\geq\ 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{BC}{HA}+\frac{CA}{HB}+\frac{AB}{HC}=\sum_{cyc}\frac{BC}{HA}=\sum_{cyc}\frac{a}{2RcosA}=\sum_{cyc}\frac{2RsinA}{2RcosA}=$$

$$=\sum_{cyc}\frac{sinA}{cosA}=\sum_{cyc}tanA\overset{JENSEN}{\geq}3tan\left(\frac{A+B+C}{3}\right)=3tan\frac{\pi}{3}=3\sqrt{3}$$

Equality holds for an equilateral triangle.

3759. If I $-$incenter in ΔABC then:

$$\frac{m_a^2}{IA^2}+\frac{m_b^2}{IB^2}+\frac{m_c^2}{IC^2}\geq\frac{27}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{m_a^2}{IA^2}+\frac{m_b^2}{IB^2}+\frac{m_c^2}{IC^2}=\sum_{cyc}\frac{m_a^2}{IA^2}=\sum_{cyc}\frac{m_a^2}{\frac{r^2}{\sin^2\frac{A}{2}}}=\frac{1}{r^2}\sum_{cyc}m_a^2\sin^2\frac{A}{2}\overset{CEBYSHEV}{\geq}$$

$$\geq\frac{1}{3r^2}\left(\sum_{cyc}m_a^2\right)\left(\sum_{cyc}\sin^2\frac{A}{2}\right)=\frac{1}{3r^2}\cdot\frac{3}{4}\sum_{cyc}a^2\cdot\left(1-\frac{r}{2R}\right)\geq$$

$$\overset{IONESCU-WEITZENBOCK}{\geq}\frac{1}{4r^2}\cdot 4\sqrt{3}F\cdot\left(1-\frac{r}{2R}\right)\overset{EULER}{\geq}\frac{1}{4r^2}\cdot 4\sqrt{3}rs\cdot\left(1-\frac{\frac{R}{2}}{2R}\right)=$$

$$=\frac{\sqrt{3}s}{r}\cdot\left(1-\frac{1}{4}\right)\overset{MITRINOVIC}{\geq}\frac{\sqrt{3}\cdot 3\sqrt{3}r}{r}\cdot\frac{3}{4}=\frac{27}{4}$$

Equality holds for an equilateral triangle.

3760. If I –incenter then in ΔABC the following relationship holds:

$$IA\cdot IB\cdot IC\leq R^3$$

Proposed by Nguyen Hung Cuong

Solution by Daniel Sitaru-Romania

$$IA\cdot IB\cdot IC=\prod_{cyc}IA=\prod_{cyc}\frac{r}{\sin\frac{A}{2}}=$$

$$=r^3\cdot\frac{1}{\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{(s-c)(s-a)}{ca}}\cdot\sqrt{\frac{(s-a)(s-b)}{ab}}}=$$

$$=\frac{r^3}{\frac{(s-a)(s-b)(s-c)}{abc}}=\frac{abcsr^3}{s(s-a)(s-b)(s-c)}=\frac{4RFsr^3}{F^2}=$$

$$=\frac{4Rsr^3}{F}=\frac{4Rsr^3}{rs}=4Rr^2\overset{EULER}{\leq}4R\cdot\frac{R^2}{4}=R^3$$

Equality holds for an equilateral triangle.

3761. In ΔABC the following relationship holds:

$$\frac{1+\sin A}{\cos A}+\frac{1+\sin B}{\cos B}+\frac{1+\sin C}{\cos C}\geq 6+3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$f:\left(0,\frac{\pi}{2}\right)\to\mathbb{R}, f(x)=\frac{1+\sin x}{\cos x}=\frac{1}{\cos x}+\tan x$$

$$f'(x)=\frac{1+\sin x}{\cos^2 x}, f''(x)=\frac{\sin^2 x+2\sin x+1}{\cos^2 x}\geq 0\Rightarrow$$

f –convexe. By Jensen's inequality:

$$\sum_{cyc}\frac{1+\sin A}{\cos A}=\sum_{cyc}f(A)\geq 3f\left(\frac{A+B+C}{3}\right)=3f\left(\frac{\pi}{3}\right)=$$

$$=3\cdot\frac{1+\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}=3\cdot\frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}}=6+3\sqrt{3}$$

Equality holds for an equilateral triangle.

3762. In ΔABC the following relationship holds:

$$2R^2(\sin^3 A+\sin^3 B+\sin^3 C)\geq 3F$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$2R^2(\sin^3 A+\sin^3 B+\sin^3 C)=2R^2\sum_{cyc}\sin^3 A=$$

$$=2R^2\sum_{cyc}\left(\frac{a}{2R}\right)^3=\frac{2R^2}{8R^3}\sum_{cyc}a^3\overset{AM-GM}{\geq}\frac{1}{4R}\cdot 3\sqrt[3]{(abc)^3}=$$

$$=\frac{3abc}{4R}=\frac{3\cdot 4RF}{4R}=3F$$

Equality holds for an equilateral triangle.

3763. In ΔABC the following relationship holds:

$$\frac{\sqrt{1+\tan^2\frac{A}{2}}}{\sin\frac{A}{2}}+\frac{\sqrt{1+\tan^2\frac{B}{2}}}{\sin\frac{B}{2}}+\frac{\sqrt{1+\tan^2\frac{C}{2}}}{\sin\frac{C}{2}}\geq 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{\sqrt{1+\tan^2\frac{A}{2}}}{\sin\frac{A}{2}}+\frac{\sqrt{1+\tan^2\frac{B}{2}}}{\sin\frac{B}{2}}+\frac{\sqrt{1+\tan^2\frac{C}{2}}}{\sin\frac{C}{2}}=\sum_{cyc}\frac{\sqrt{1+\tan^2\frac{A}{2}}}{\sin\frac{A}{2}}=$$

$$=\sum_{cyc}\frac{\sqrt{\frac{1}{\cos^2\frac{A}{2}}}}{\sin\frac{A}{2}}=\sum_{cyc}\frac{1}{\sin\frac{A}{2}\cos\frac{A}{2}}=\sum_{cyc}\frac{2}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\sum_{cyc}\frac{2}{\sin A}=$$

$$=\sum_{cyc}\frac{4R}{a}=4R\sum_{cyc}\frac{1}{a}\overset{AM-GM}{\geq}4R\cdot\frac{3}{\sqrt[3]{abc}}=\frac{12R}{\sqrt[3]{4Rrs}}\overset{MITRINOVICI}{\geq}$$

$$\geq\frac{12R}{\sqrt[3]{4Rr\cdot\frac{3\sqrt{3}}{2}R}}=\frac{12R}{\sqrt{3}\cdot\sqrt[3]{2R^2r}}\overset{EULER}{\geq}\frac{12R}{\sqrt{3}\cdot\sqrt[3]{2R^2\cdot\frac{R}{2}}}=\frac{12}{\sqrt{3}}=4\sqrt{3}$$

Equality holds for an equilateral triangle.

3764. In ΔABC the following relationship holds:

$$\frac{27r}{R}\leq\sum r_a\sin^2 A\leq\frac{27R^2}{16r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\textit{Known result}:\sum\frac{a^2}{s-a}=\frac{4s}{r}(R-r)$$

$$\sum r_a\sin^2 A=\frac{1}{4R^2}\sum a^2.\frac{F}{s-a}=\frac{F}{4R^2}\sum\frac{a^2}{s-a}=\frac{F}{4R^2}\cdot\frac{4s}{r}(R-r)=\frac{s^2}{R^2}(R-r)$$

We need to show :

$$\frac{s^2}{R^2}(R-r)\le\frac{27R^2}{16r}\ or\ 16s^2r(R-r)\le 27R^4$$

$$16\frac{27}{4}R^2.r(R-r)\overset{Mitrinovic}{\le}27R^4\ or\ R^2-4Rr+4r^2\ge 0\ or\ (R-2r)^2\ge 0\ true$$

$$We\ need\ to\ show:$$

$$\frac{s^2}{R^2}(R-r)\ge\frac{27r}{R}\ or,\frac{27Rr}{2R^2}(R-r)\overset{s^2\ge\frac{27Rr}{2}}{\ge}\frac{27r}{R}\ or,2(R-r)\ge R\ or,R\ge 2r\ true$$

Equality holds for an equilateral triangle.

3765. In ΔABC the following relationship holds:

$$\frac{3}{s}\le\sum\frac{AH}{ah_a}\le\frac{3R}{2rs}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{AH}{ah_a}=\sum\frac{2RcosA}{a\frac{2F}{a}}=\frac{R}{F}\sum cosA=\frac{R}{r.s}\left(1+\frac{r}{R}\right)\overset{Euler}{\le}\frac{3R}{2rs}$$

$$\sum\frac{AH}{ah_a}=\sum\frac{2RcosA}{a\frac{2F}{a}}=\frac{R}{F}\sum cosA=\frac{R}{r.s}\left(1+\frac{r}{R}\right)=\frac{R+r}{r.s}\overset{Euler}{\ge}\frac{2r+r}{r.s}=\frac{3}{s}$$

Equality holds for an equilateral triangle.

3766. In ΔABC the following relationship holds:

$$\frac{27r^2}{2R}\le\sum h_a\sin^2 A\le\frac{27r}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum h_a\sin^2 A=2F\sum\frac{sinA}{a}.sinA=\frac{F}{R}\sum sinA=\frac{F}{R}\cdot\frac{s}{R}=\frac{s^2r}{R^2}$$

$$\sum h_a \sin^2 A = \frac{s^2 r}{R^2} \overset{Mitrinovic}{\leq} \frac{27}{4} R^2 . \frac{r}{R^2} = \frac{27r}{4}$$

$$\sum h_a \sin^2 A = \frac{s^2 r}{R^2} \overset{s^2 \geq \frac{27Rr}{2}}{\geq} \frac{27Rr}{2} . \frac{r}{R^2} = \frac{27r^2}{2R}$$

Equality holds for an equilateral triangle.

3767. **In ΔABC the following relationship holds:**

$$\sum \frac{AH}{ah_a} \leq \sum \frac{AH}{ar_a}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\textit{Known results}: 1) AH = 2R\cos A, 2) r_a = s\tan\frac{A}{2}$$

$$3) \sum \tan^2\frac{A}{2}.\tan^2\frac{B}{2} = \frac{s^2 - 2r^2 - 8Rr}{s^2} \quad 4) \prod \tan^2\frac{A}{2} = \left(\frac{r}{s}\right)^2$$

$$\sum \cot^2\frac{A}{2} = \frac{\sum \tan^2\frac{A}{2}.\tan^2\frac{B}{2}}{\prod \tan^2\frac{A}{2}} = \frac{s^2 - 2r^2 - 8Rr}{r^2}$$

$$\frac{AH}{ar_a} = \frac{2R\cos A}{2R\sin A . s\tan\frac{A}{2}} = \frac{\cot A}{s\tan\frac{A}{2}} = \frac{\frac{\cot^2\frac{A}{2} - 1}{2\cot\frac{A}{2}}}{.s\tan\frac{A}{2}} = \frac{1}{2s}\left(\cot^2\frac{A}{2} - 1\right)$$

$$\sum \frac{AH}{ar_a} = \frac{1}{2s}\sum\left(\cot^2\frac{A}{2} - 1\right) = \frac{1}{2s}\left(\frac{s^2 - 2r^2 - 8Rr}{r^2} - 3\right) =$$

$$= \frac{1}{2s}\frac{s^2 - 5r^2 - 8Rr}{r^2} \overset{Gerrestn}{\geq} \frac{1}{2s}\frac{16Rr - 5r^2 - 5r^2 - 8Rr}{r^2} = \frac{1}{s}\left(\frac{4R}{r} - 5\right)$$

$$\sum \frac{AH}{ah_a} = \sum \frac{2R\cos A}{a.\frac{2F}{a}} = \frac{R}{F}\sum \cos A = \frac{R}{r.s}\left(1 + \frac{R}{r}\right) \overset{Euler}{\leq} \frac{3R}{2rs}$$

$$\textit{We need to show}:$$

$$\frac{1}{s}\left(\frac{4R}{r} - 5\right) \geq \frac{3R}{2rs} \quad or, \frac{4R}{r} - \frac{3R}{2r} \geq 5 \; or, \frac{5R}{2r} \geq 5 \; or, R \geq 2r \; \textit{true by Euler}.$$

Equality holds for an equilateral triangle.

3768. In any Δ ABC the following relationship holds :

$$\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}\ge\frac{1}{5}\cdot\sqrt{\frac{64R}{r}+97}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{p_a^2-h_a^2}{h_a^2}\overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ben Ajiba}}}{=}\frac{s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-\left(s(s-a)-\frac{s(s-a)}{a^2}(b-c)^2\right)}{h_a^2}$$

$$=\frac{\frac{4s^4}{a^2(2s+a)^2}}{\frac{4r^2s^2}{a^2}}.(b-c)^2\therefore\frac{p_a^2}{h_a^2}\overset{①}{=}1+\frac{s^2}{r^2}\cdot\frac{(b-c)^2}{(2s+a)^2}\text{ and analogs}\therefore\sum_{cyc}\frac{p_a^2}{h_a^2}\overset{②}{=}3+$$

$$\frac{s^2}{r^2}\cdot\left(\left(\sum_{cyc}\frac{b-c}{2s+a}\right)^2-\frac{2}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc}\big((b-c)(c-a)(2s+c)\big)\right)$$

$$\text{Now,}\left(\sum_{cyc}\frac{b-c}{2s+a}\right)^2=\frac{1}{4s^2(9s^2+6Rr+r^2)^2}\cdot\left(\sum_{cyc}\big((b-c)(8s^2-2sa+bc)\big)\right)^2$$

$$=\frac{1}{4s^2(9s^2+6Rr+r^2)^2}\cdot\left(\sum_{cyc}b^2c-\sum_{cyc}bc^2\right)^2$$

$$=\frac{(\sum_{cyc}a^2)(\sum_{cyc}a^2b^2)-15a^2b^2c^2-2abc\sum_{cyc}a^3-2\sum_{cyc}a^3b^3+2abc(\sum_{cyc}a)(\sum_{cyc}ab)}{4s^2(9s^2+6Rr+r^2)^2}$$

$$\overset{(*)}{=}\left(\sum_{cyc}\frac{b-c}{2s+a}\right)^2\text{ and also,}-\frac{2}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc}\big((b-c)(c-a)(2s+c)\big)\overset{(**)}{=}$$

$$-2.\frac{2s(\sum_{cyc}ab-\sum_{cyc}a^2)+2s(\sum_{cyc}ab)-\sum_{cyc}a^3-24Rrs}{2s(9s^2+6Rr+r^2)}\text{ and since : }\sum_{cyc}ab=$$

$$s^2+4Rr+r^2,\sum_{cyc}a^2=2(s^2-4Rr-r^2),\sum_{cyc}a^2b^2=(s^2+4Rr+r^2)^2-16Rrs^2,$$

$$\sum_{cyc}a^3=2(s^2-6Rr-3r^2),\sum_{cyc}a^3b^3=(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)$$

$$\therefore ②,(*),(**) \Rightarrow \sum_{cyc}\frac{p_a^2}{h_a^2} \boxed{\overset{(m)}{=}} 3+\frac{18s^6-(168Rr+125r^2)s^4-r^2(116R^2+84Rr+16r^2)s^2-r^3(4R+r)^3}{r^2(9s^2+6Rr+r^2)^2}$$

$$\textit{and again}, ① \Rightarrow \sum_{cyc}\frac{p_bp_c}{h_bh_c}=\sum_{cyc}\sqrt{\left(1+\frac{s^2}{r^2}\cdot\frac{(c-a)^2}{(2s+b)^2}\right)\cdot\left(1+\frac{s^2}{r^2}\cdot\frac{(a-b)^2}{(2s+c)^2}\right)}$$

$$\overset{\text{Reverse CBS}}{\geq} 3+\frac{s^2}{r^2}\cdot\frac{1}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc}(|c-a||a-b|(2s+a)) \overset{\text{Triangle Inequality}}{\geq}$$

$$3+\frac{s^2}{r^2}\cdot\frac{1}{2s(9s^2+6Rr+r^2)}\cdot\left(2s\left|\sum_{cyc}(c-a)(a-b)\right|+\left|\sum_{cyc}a(c-a)(a-b)\right|\right)$$

$$=3+\frac{s^2\left((s^2-12Rr-3r^2)+(2Rr-4r^2)\right)}{r^2(9s^2+6Rr+r^2)} \therefore 2\sum_{cyc}\frac{p_bp_c}{h_bh_c}+\sum_{cyc}\frac{p_a^2}{h_a^2}\geq 9+$$

$$\frac{s^2(9s^2+6Rr+r^2)(2s^2-20Rr-14r^2)+18s^6-(168Rr+125r^2)s^4-r^2(116R^2+84Rr+16r^2)s^2-r^3(4R+r)^3}{r^2(9s^2+6Rr+r^2)^2}\overset{?}{\geq}\frac{64R+97r}{25r}$$

$$\Leftrightarrow 900s^6-(13584Rr-4143r^2)s^4-r^2(12812R^2-7972Rr-1554r^2)s^2-$$
$$r^3(3904R^3-2640R^2r-1172Rr^2-103r^3) \boxed{\overset{(\bullet)}{\geq}} 0; \because P=900(s^2-16Rr+5r^2)^3$$
$$+(29616Rr-9357r^2)(s^2-16Rr+5r^2)^2+$$
$$4r^2(60925R^2-38903Rr+6906r^2)\left(s^2-16Rr+5r^2-\frac{r^2(R-2r)}{R-r}\right) \overset{\substack{\text{Gerretsen}\\ \text{and Rouche}\\ +\text{Euler}}}{\geq} 0$$

$$\left(\begin{array}{c}\because (R-r)(s^2-16Rr+5r^2) \overset{\text{Rouche}}{\geq} \\ (R-r)\left(2R^2-6Rr+4r^2-2(R-2r)\sqrt{R^2-2Rr}\right) \\ =(R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2+r^2\right)\geq r^2(R-2r)\ \left(\because R-2r \overset{\text{Euler}}{\geq} 0\right) \\ \Rightarrow s^2\geq 16Rr-5r^2+\dfrac{r^2(R-2r)}{R-r}\end{array}\right)$$

$\therefore$ in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet) \overset{?}{\geq} P$

$$\Leftrightarrow (t-2)(3375t^2-1869t+302)\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet) \text{ is true}$$

$$\therefore \left(\sum_{cyc}\frac{p_a}{h_a}\right)^2\geq\frac{64R+97r}{25r}\Rightarrow\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}\geq\frac{1}{5}\cdot\sqrt{\frac{64R}{r}+97}\ \forall\,\Delta\,ABC,$$

$''=''$ iff Δ ABC is equilateral (QED)

3769. In any acute Δ ABC the following relationship holds :

$$2\left(\frac{1}{\cos A}+\frac{1}{\cos B}\right)\geq\sqrt{3}(\tan A+\tan B)+2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\Delta\ \text{ABC being acute}\Rightarrow\cos A,\cos B>0$$
$$\therefore 2\left(\frac{1}{\cos A}+\frac{1}{\cos B}\right)\overset{?}{\geq}\sqrt{3}.(\tan A+\tan B)+2$$
$$\Leftrightarrow 2(\cos A+\cos B)\overset{?}{\geq}\sqrt{3}.(\sin A\cos B+\cos A\sin B)+2\cos A\cos B$$
$$=\sqrt{3}.\sin C+\cos(A+B)+\cos(A-B)$$
$$\Leftrightarrow 4\sin\frac{C}{2}\cos\frac{A-B}{2}\overset{?}{\geq}\sqrt{3}.\sin C-\cos C+2\cos^2\frac{A-B}{2}-1$$
$$\Leftrightarrow 2t^2-4\sin\frac{C}{2}.t+\sqrt{3}.\sin C-2\cos^2\frac{C}{2}\underset{(*)}{\overset{?}{\leq}}0\ \left(t=\cos\frac{A-B}{2}\right)\text{ and to prove }(*),$$

$$\text{it suffices to prove : } t\ \boxed{\underset{①}{\overset{?}{\leq}}}\ \frac{4\sin\frac{C}{2}+\sqrt{16\sin^2\frac{C}{2}-8\left(\sqrt{3}.\sin C-2\cos^2\frac{C}{2}\right)}}{4}\text{ AND}$$

$$t\ \boxed{\underset{②}{\overset{?}{\geq}}}\ \frac{4\sin\frac{C}{2}-\sqrt{16\sin^2\frac{C}{2}-8\left(\sqrt{3}.\sin C-2\cos^2\frac{C}{2}\right)}}{4};\because t\leq 1\therefore\text{in order to prove } ①,$$

$$\text{it suffices to prove : }1\overset{?}{\leq}\sin\frac{C}{2}+\sqrt{1-\frac{\sqrt{3}}{2}.\sin C}\Leftrightarrow\left(1-\sin\frac{C}{2}\right)^2\overset{?}{\leq}1-\frac{\sqrt{3}}{2}.\sin C$$
$$\Leftrightarrow\sin^2\frac{C}{2}-2\sin\frac{C}{2}+\sqrt{3}.\sin\frac{C}{2}\cos\frac{C}{2}\overset{?}{\leq}0\Leftrightarrow\sin\frac{C}{2}+\sqrt{3}.\cos\frac{C}{2}\overset{?}{\leq}2\rightarrow\text{true}$$
$$\because\sin\frac{C}{2}+\sqrt{3}.\cos\frac{C}{2}\overset{\text{CBS}}{\leq}\sqrt{1+3}.\sqrt{\sin^2\frac{C}{2}+\cos^2\frac{C}{2}}=2\Rightarrow ①\text{ is true with equality iff}$$
$$\frac{C}{2}=\frac{\pi}{6}\wedge A=B\Rightarrow\text{equality iff }A=B=C=\frac{\pi}{3}\text{ and again, }0<A,B<\frac{\pi}{2}\Rightarrow$$
$$-\frac{\pi}{4}<\frac{A-B}{2}<\frac{\pi}{4}\Rightarrow t\geq\frac{1}{\sqrt{2}}\overset{?}{>}\sin\frac{C}{2}-\sqrt{1-\frac{\sqrt{3}}{2}.\sin C}$$
$$\Leftrightarrow\frac{1}{2}+1-\frac{\sqrt{3}}{2}.\sin C+2.\sqrt{1-\frac{\sqrt{3}}{2}.\sin C}\overset{?}{>}\sin^2\frac{C}{2}\text{ and }\because\sqrt{1-\frac{\sqrt{3}}{2}.\sin C}>0$$
$$\therefore\text{it suffices to prove : }3-\sqrt{3}.\sin C\overset{?}{>}1-\cos C\Leftrightarrow\cos\frac{\pi}{6}\sin C-\sin\frac{\pi}{6}\cos C\overset{?}{<}1$$

$$\Leftrightarrow \sin\left(C-\frac{\pi}{6}\right)\overset{?}{<}1\to \text{true}\Rightarrow ② \text{ is true (strict inequality) and so, } ① \text{ and } ②$$

$$\text{are true}\Rightarrow (*) \text{ is true} \therefore 2\left(\frac{1}{\cos A}+\frac{1}{\cos B}\right)\geq\sqrt{3}.(\tan A+\tan B)+2$$

$$\forall \text{ acute } \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$f(x)=\frac{1}{\cos(x)}-\sqrt{3}\tan(x)\to\ f(A)+f(B)\geq 2$$

$$\textit{Investigation of extreme values}:$$

$$f(x)=2\sec(x)-\sqrt{3}\tan(x)\to\frac{d}{dx}f(x)=2\sec(x)\tan(x)-\sqrt{3}\sec^2(x)$$

$$\frac{d}{dx}f(x)=\frac{2\sin(x)-\sqrt{3}}{\cos^2(x)}\quad \frac{d}{dx}f(x)=0\to\sin(x)=\frac{\sqrt{3}}{2}$$

$$\textit{The unique root in the acute interval}$$

$$\left(0;\frac{\pi}{2}\right)\ is\ x=\frac{\pi}{3}$$

$$\textit{Characterization of the function and global minimum}$$

- $x\in\left(0;\frac{\pi}{3}\right)\to\sin(x)<\frac{\sqrt{3}}{2}\quad \frac{d}{dx}f(x)<0\ \ \textit{decreasing function}$
- $x\in\left(\frac{\pi}{3};\frac{\pi}{2}\right)\to\sin(x)>\frac{\sqrt{3}}{2}\quad \frac{d}{dx}f(x)>0\ \ \textit{increasing function}$

$$\textit{The minimum point of the function } x=\frac{\pi}{3}$$

$$f\left(\frac{\pi}{3}\right)=\frac{1}{\cos\left(\frac{\pi}{3}\right)}-\sqrt{3}\tan\left(\frac{\pi}{3}\right)=1$$

$$x\in\left(0;\frac{\pi}{2}\right)\to\ f(x)\geq 1\ \to\begin{cases}f(A)\geq 1\\ f(B)\geq 1\end{cases}\ f(A)+f(B)\geq 2$$

$$2\left(\frac{1}{\cos(A)}+\frac{1}{\cos(B)}\right)\geq\sqrt{3}(\tan(A)+\tan(B))+2$$

3770. In ΔABC the following relationship holds:

$$\sum_{cyc}\csc\left(\frac{A}{2}\right)\left(1+\sin\left(\frac{A}{2}\right)\right)\geq 9$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Amin Hajiyev-Azerbaijan

$$\sum_{cyc}\csc\left(\frac{A}{2}\right)\left(1+\sin\left(\frac{A}{2}\right)\right)\geq 9$$

$$\sum_{cyc}\csc\left(\frac{A}{2}\right)+\sum_{cyc}\csc\left(\frac{A}{2}\right)\sin\left(\frac{A}{2}\right)\geq 9\to\sum_{cyc}\csc\left(\frac{A}{2}\right)+3\geq 9$$

$$\sum_{cyc}\csc\left(\frac{A}{2}\right)\geq 6\to A+B+C=\pi\quad A\in(0;\ \pi)\to\csc\left(\frac{A}{2}\right)>0$$

$$f(x)=\csc\left(\frac{x}{2}\right)\to\frac{d^2}{dx^2}f(x)=\frac{1}{4}\csc\left(\frac{x}{2}\right)\left(csc^2\left(\frac{x}{2}\right)+ctg^2\left(\frac{x}{2}\right)\right)$$

$$x\in(0;\pi)\to\frac{d^2}{dx^2}f(x)>0$$

$$\frac{1}{3}\sum_{cyc}\csc\left(\frac{A}{2}\right)\overset{JENSEN}{\geq}\csc\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)\to\sum_{cyc}\csc\left(\frac{A}{2}\right)\geq 3\csc\left(\frac{\pi}{6}\right)=6$$

$$\sum_{cyc}\csc\left(\frac{A}{2}\right)\left(1+\sin\left(\frac{A}{2}\right)\right)\geq 9$$

Equality holds for an equilateral triangle.

3771. In ΔABC the following relationship holds:

$$\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}+\frac{\csc\left(\frac{B}{2}\right)}{1+\cos\left(\frac{B}{2}\right)}+\frac{\csc\left(\frac{C}{2}\right)}{1+\cos\left(\frac{C}{2}\right)}\geq\frac{12}{2+\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Amin Hajiyev-Azerbaijan

$$\sum_{cyc}\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}\geq\frac{12}{2+\sqrt{3}}\to LHS=\sum_{cyc}\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}=\sum_{cyc}\frac{1}{\sin\left(\frac{A}{2}\right)\left(1+\cos\left(\frac{A}{2}\right)\right)}$$

$$A+B+C=\pi\to\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}\quad x\in\left(0;\frac{\pi}{2}\right)\to\begin{cases}0<\sin(x)<1\\0<\cos(x)<1\end{cases}$$

$$f(x)=\frac{1}{\sin(x)(1+\cos(x))}\qquad\frac{d^2}{dx}f(x)>0\to Convexity$$

$$Jensen's\ inequality:\ \frac{\sum_{i=1}^{n}f(x_i)}{n}\geq f\left(\sum_{i=1}^{n}x_i\right)$$

$$\frac{1}{3}\sum_{cyc}\frac{1}{\sin\left(\frac{A}{2}\right)\left(1+\cos\left(\frac{A}{2}\right)\right)}\geq f\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)=f\left(\frac{\pi}{6}\right)$$

$$\sum_{cyc}\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}\geq\frac{3}{\sin\left(\frac{\pi}{6}\right)\left(1+\cos\left(\frac{\pi}{6}\right)\right)}=\frac{3}{\frac{1}{2}\left(1+\frac{\sqrt{3}}{2}\right)}$$

$$\sum_{cyc}\frac{\csc\left(\frac{A}{2}\right)}{1+\cos\left(\frac{A}{2}\right)}\ge\frac{12}{2+\sqrt{3}}$$

Equality holds for an equilateral triangle.

3772. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{cotB+cotC}{sinA}\ge 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{cot(B)+cot(C)}{\sin(A)}=\frac{\frac{\cos(B)}{\sin(B)}+\frac{\cos(C)}{\sin(C)}}{\sin(A)}=\frac{\sin(B)\cos(C)+\sin(C)\cos(B)}{\sin(A).\sin(B).\sin(C)}=$$

$$=\frac{\sin(B+C)}{\sin(A).\sin(B).\sin(C)}=\frac{1}{\sin(B).\sin(C)}$$

$$\sum_{cyc}\frac{1}{\sin(B).\sin(C)}=\frac{\sin(A)+\sin(B)+\sin(C)}{\sin(A).\sin(B).\sin(C)}=\frac{\frac{s}{R}}{\frac{F}{2R^2}}=\frac{2sR^2}{FR}=\frac{2sR}{sr}=\frac{2R}{r}\overset{Euler}{\ge}4$$

$$Equality\ \ holds\ \ for: A=B=C$$

3773. In ΔABC the following relationship holds:

$$\frac{\sin(A)}{\tan(B)+\tan(C)}+\frac{\sin(B)}{\tan(C)+\tan(A)}+\frac{\sin(C)}{\tan(A)+\tan(B)}\ge\frac{3}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{\sin(A)}{\tan(B)+\tan(C)}=\sum_{cyc}\frac{\sin(A)}{\frac{\sin(B+C)}{\cos(B).\cos(C)}}=$$

$$=\sum_{cyc}\cos(A).\cos(B)=\frac{s^2-r^2-4R^2}{4R^2}\overset{Gerretsen}{\le}$$

$$\le\frac{4R^2+4Rr+3r^2+r^2-R^2}{4R^2}=\frac{Rr+r^2}{R^2}\overset{Euler}{\le}\frac{\frac{R^2}{2}+\frac{R^2}{4}}{R^2}=\frac{3}{4}$$

$$Equality\ \ holds\ \ for: A=B=C.$$

3774. Let P be the Brocard's point of acute – angled ABC. R_a, R_b, R_c circumradius of BPC, APC, APB. H – orthocenter. Prove that:

$$\frac{R_a \bullet h_a}{AH} + \frac{R_b \bullet h_b}{BH} + \frac{R_c \bullet h_c}{CH} \geq 9r$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

Lemma 1: $\cos A \bullet \cos B \bullet \cos C \leq \frac{1}{8}$

Lemma 2: $h_a = 2R \bullet \sin B \bullet \sin C$

$$\frac{a}{\sin(180 - C)} = 2R_a \Rightarrow R_a = \frac{a}{2\sin C}$$

$$\text{LHS} = \sum \frac{R_a \bullet h_a}{AH} = \sum \frac{\frac{a}{2\sin C} \bullet 2R\sin B\sin C}{2R\cos A} = \sum \frac{a\sin B}{2\cos A} \geq \frac{3}{2}\sqrt[3]{\frac{abc\prod \sin A}{\prod \cos A}}$$

$$\geq 3\sqrt[3]{abc\prod \sin A} = 3\sqrt[3]{abc\left(\frac{abc}{8R^3}\right)} = 3\sqrt[3]{\frac{(abc)^2}{8R^3}} = 3\sqrt[3]{\frac{16R^2r^2s^2}{8R^3}}$$

$$= 3\sqrt[3]{\frac{2s^2r^2}{R}} \geq 3\sqrt[3]{\left(\frac{27Rrr^2}{R}\right)} = 9r$$

Equality holds for a=b=c.

3775. In any Δ ABC the following relationship holds :

$$\frac{n_a^2 + g_a^2}{rh_a} + \frac{n_a}{n_a - \sqrt{4r^2 + (b-c)^2}} \leq 1 + \frac{4R}{r}\left(\frac{r_b + r_c}{h_a} - 1\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c - h_a \overset{?}{\geq} \frac{b^2 + c^2}{4R} \Leftrightarrow 4R\cos^2\frac{A}{2} \overset{?}{\geq} \frac{(b+c)^2}{4R}$$

$$\Leftrightarrow 4R\cos\frac{A}{2} \overset{?}{\geq} 4R\cos\frac{A}{2}\cos\frac{B-C}{2} \to \text{true} \because \cos\frac{B-C}{2} \leq 1 \therefore r_b + r_c - h_a \geq \frac{b^2 + c^2}{4R}$$

$$\Rightarrow \frac{4R}{r}\left(\frac{r_b + r_c}{h_a} - 1\right) \geq \frac{4R}{rh_a} \cdot \frac{b^2 + c^2}{4R} = \frac{b^2 + c^2}{rh_a}$$

$$\Rightarrow \frac{n_a^2 + g_a^2}{rh_a} + \frac{n_a}{n_a - \sqrt{4r^2 + (b-c)^2}} - 1 - \frac{4R}{r}\left(\frac{r_b + r_c}{h_a} - 1\right)$$

$$\overset{\text{Bogdan Fustei}}{=}\frac{(b-c)^2+2s(s-a)}{rh_a}+\frac{n_a}{n_a-\frac{an_a}{s}}-1-\frac{b^2+c^2}{rh_a}$$

$$=\frac{b^2+c^2}{rh_a}-\frac{2R.2bc}{r.bc}+\frac{2s(s-a)a}{2(s-a)(s-b)(s-c)}+\frac{s}{s-a}-1-\frac{b^2+c^2}{rh_a}$$

$$=-\frac{4R}{r}+\frac{sa}{(s-b)(s-c)}+\frac{a}{s-a}=-\frac{4R}{r}+\frac{a}{r^2s}.\left(s(s-a)+(s-b)(s-c)\right)$$

$$=-\frac{4R}{r}+\frac{a}{r^2s}.(s^2-sa-s^2+sa+bc)=-\frac{4R}{r}+\frac{4Rrs}{r^2s}=0$$

$$\therefore\frac{n_a^2+g_a^2}{rh_a}+\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\le 1+\frac{4R}{r}\left(\frac{r_b+r_c}{h_a}-1\right)\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } b=c\ (QED)$$

3776. In any Δ ABC the following relationship holds :

$$3+\sum_{cyc}\frac{b+c}{\sqrt{4r^2+(b-c)^2}}>\frac{1}{r}\sum_{cyc}h_a$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$3+\sum_{cyc}\frac{b+c}{\sqrt{4r^2+(b-c)^2}}\overset{\text{Bogdan Fustei}}{=}3+\sum_{cyc}\frac{s(b+c)}{an_a}$$

$$=3+\sum_{cyc}\frac{\sqrt{s^2-2h_ar_a+2h_ar_a}.(b+c)}{an_a}\overset{\text{Bogdan Fustei}}{=}3+\sum_{cyc}\frac{\sqrt{n_a^2+2h_ar_a}.(b+c)}{an_a}$$

$$>3+\sum_{cyc}\frac{n_a.(b+c)}{an_a}=\sum_{cyc}\left(1+\frac{b+c}{a}\right)=\frac{1}{r}\sum_{cyc}\frac{2rs}{a}=\frac{1}{r}\sum_{cyc}h_a$$

$$\therefore 3+\sum_{cyc}\frac{b+c}{\sqrt{4r^2+(b-c)^2}}>\frac{1}{r}\sum_{cyc}h_a\ \forall\,\Delta\,ABC\ (QED)$$

3777. If I_a,I_b,I_c – excenters in ΔABC then prove that:

$$\frac{2}{F}\left(2-\frac{r}{R}\right)\le\sum_{cyc}\frac{1}{F_{BCI_a}}\le\frac{2}{F}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Proposed by Eldeniz Hesenov-Georgia

Solution by Tapas Das-India

$$F_{BCI_a} = \frac{1}{2}.r_a.a = \frac{F}{2}\frac{a}{s-a}$$

$$\sum_{cyc}\frac{1}{F_{BCI_a}} = \frac{2}{F}\sum\frac{s-a}{a} = \frac{2}{F}\left(s\sum\frac{1}{a}-3\right) = \frac{2}{F}\left(s\frac{ab+bc+ca}{abc}-3\right) =$$

$$= \frac{2}{F}\left(\frac{s^2+r^2+4Rr}{4Rr}-3\right) = \frac{2}{F}\left(\frac{s^2+r^2-8Rr}{4Rr}\right) \overset{Gerretsen}{\geq}$$

$$\geq \frac{2}{F}\left(\frac{16Rr-5r^2+r^2-8Rr}{4Rr}\right) = \frac{2}{F}\left(\frac{8Rr-4r^2}{4Rr}\right) = \frac{2}{F}\left(2-\frac{r}{R}\right)$$

$$\sum_{cyc}\frac{1}{F_{BCI_a}} = \frac{2}{F}\left(\frac{s^2+r^2-8Rr}{4Rr}\right) \overset{Gerretsen}{\leq} \frac{2}{F}\left(\frac{4R^2+4Rr+3r^2+r^2-8Rr}{4Rr}\right) =$$

$$= \frac{2}{F}\left(\frac{4R^2-4Rr+4r^2}{4Rr}\right) = \frac{2}{F}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Equality holds for an equilateral triangle.

3778. In ΔABC the following relationship holds:

$$\frac{r_a^4h_a}{m_a+h_am_a^2}+\frac{r_b^4h_b}{m_b+h_bm_b^2}+\frac{r_c^4h_c}{m_c+h_cm_c^2} \geq \frac{1944r^5}{4R+9R^3}$$

Proposed by Elsen Kerimov, Shirvan Tahirov-Azerbaijan

Solution by Tapas Das-India

$$\sum\frac{m_a}{h_a} \overset{Chebyshev}{\leq} \frac{1}{3}\left(\sum m_a\right)\left(\sum\frac{1}{h_a}\right) \overset{Gotman}{\leq} \frac{1}{3}.\frac{9R}{2}.\frac{1}{r} = \frac{3R}{2r},$$

$$\sum m_a^2 = \frac{3}{4}\left(\sum a^2\right) \overset{Leibniz}{\leq} \frac{27R^2}{4}$$

$$\sum r_a^2 = \left(\sum r_a\right)^2 - 2\sum r_ar_b = (4R+r)^2-2s^2 \overset{Doucet}{\geq}$$

$$\geq 3s^2-2s^2 = s^2 \overset{Mitrinovic}{\geq} 27r^2$$

$$\frac{r_a^4h_a}{m_a+h_am_a^2}+\frac{r_b^4h_b}{m_b+h_bm_b^2}+\frac{r_c^4h_c}{m_c+h_cm_c^2}=\sum\frac{r_a^4h_a}{m_a+h_am_a^2}=$$

$$=\sum\frac{r_a^4}{\frac{m_a}{h_a}+m_a^2}\overset{Bergstrom}{\geq}\frac{(\sum r_a^2)^2}{\sum\frac{m_a}{h_a}+\sum m_a^2}\geq\frac{(27r^2)^2}{\frac{3R}{2r}+\frac{27R^2}{4}}=\frac{243r^5\times 4}{2R+9R^2r}\overset{Euler}{\geq}$$

$$\geq\frac{243r^5\times 4}{2R+\frac{9R^2R}{2}}=\frac{1944r^5}{4R+9R^3}$$

Equality holds for an equilateral triangle.

3779. In ΔABC the following relationship holds:

$$\frac{3r}{R}\leq\sum_{cyc}\frac{a^2}{b^2+c^2}\leq\frac{2R-r}{2r}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{AM-GM}{\overbrace{\leq}}\sum_{cyc}\frac{a^2}{2bc}=\sum_{cyc}\frac{a^3}{2abc}=$$

$$=\frac{2(s^3-3sr^2-6sRr)}{2.4Rsr}=\frac{s^2-3r^2-6Rr}{4Rr}\overset{Gerretsen}{\overbrace{\leq}}$$

$$\leq\frac{4R^2+4Rr+3r^2-3r^2-6Rr}{4Rr}=\frac{4R^2-2Rr}{4Rr}=\frac{2R-r}{2r}\quad(*)$$

$$\sum_{cyc}\frac{a^2}{b^2+c^2}\overset{Nesbitt}{\overbrace{\geq}}\frac{3}{2}\overset{Euler}{\overbrace{\geq}}\frac{3r}{R}\quad(**)$$

$$From\ (*)\ and\ (**)\ we\ have:$$

$$\frac{3r}{R}\leq\sum_{cyc}\frac{a^2}{b^2+c^2}\leq\frac{2R-r}{2r}$$

$$Equality\ holds\ for: a=b=c$$

3780. In ΔABC the following relationship holds:

$$\sum_{cyc}\left(a^2\tan\left(\frac{B}{2}\right).\tan\left(\frac{C}{2}\right)\right)^n\geq 3(4r^2)^n,\ \ n\in N$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$a^2\tan\left(\frac{B}{2}\right).\tan\left(\frac{C}{2}\right)=a^2\frac{\prod_{cyc}\tan\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right)}=a^2.\frac{r}{s}.\frac{1}{\tan\left(\frac{A}{2}\right)}$$

$$\sum_{cyc}\left(a^2\tan\left(\frac{B}{2}\right).\tan\left(\frac{C}{2}\right)\right)^n=\sum_{cyc}\left(a^2.\frac{r}{s}.\frac{1}{\tan\left(\frac{A}{2}\right)}\right)^n\overset{AM-GM}{\geq}$$

$$\geq 3\left((abc)^2.\left(\frac{r}{s}\right)^3.\frac{1}{\prod\tan\left(\frac{A}{2}\right)}\right)^{\frac{n}{3}}=3\left((4RF)^2.\left(\frac{r}{s}\right)^3.\frac{1}{\left(\frac{r}{s}\right)}\right)^{\frac{n}{3}}=$$

$$=3\left(16R^2F^2.\left(\frac{r}{s}\right)^2\right)^{\frac{n}{3}}=3\left(16R^2.s^2r^2.\left(\frac{r}{s}\right)^2\right)^{\frac{n}{3}}\overset{Euler}{\geq}3(16.4r^2.r^2.r^2)^{\frac{n}{3}}=3(4r^2)^n$$

$$Equality\ \ holds\ \ for: a=b=c$$

3781. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{2r_a^2-9r^2}{r_a}\geq 9r$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{2r_a^2-9r^2}{r_a}=2\sum_{cyc}r_a-9r^2\sum_{cyc}\frac{1}{r_a}=2(4R+r)-9r^2.\frac{1}{r}=$$

$$=8R-7r\overset{Euler}{\geq}16r-7r=9r$$

$$Equality\ \ holds\ \ for: a=b=c.$$

3782. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a}{b+c}+\lambda\prod_{cyc}\frac{w_a}{r_a}\geq\lambda+\frac{3}{2},\lambda\leq\frac{1}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$w_a \le \sqrt{r_b r_b} \text{ and analogs} \Rightarrow \prod_{cyc} \frac{w_a}{r_a} \le 1 \therefore \sum_{cyc} \frac{a}{b+c} - \frac{3}{2} - \lambda\left(1 - \prod_{cyc} \frac{w_a}{r_a}\right)$$

$$\overset{\lambda \le \frac{1}{2}}{\ge} \sum_{cyc} \frac{a\left(a^2 + \sum_{cyc} ab\right)}{2s(s^2+2Rr+r^2)} - \frac{3}{2} - \frac{1}{2}\left(1 - \frac{16Rr^2s^2}{(s^2+2Rr+r^2)rs^2}\right)$$

$$= \frac{2s(s^2-6Rr-3r^2) + 2s(s^2+4Rr+r^2) - 3s(s^2+2Rr+r^2)}{2s(s^2+2Rr+r^2)} -$$

$$\frac{1}{2} \cdot \frac{s^2-14Rr+r^2}{s^2+2Rr+r^2} = \frac{s^2-10Rr-7r^2}{2(s^2+2Rr+r^2)} - \frac{s^2-14Rr+r^2}{2(s^2+2Rr+r^2)} = \frac{4r(R-2r)}{2(s^2+2Rr+r^2)}$$

$$\overset{\text{Euler}}{\ge} 0 \therefore \sum_{cyc} \frac{a}{b+c} + \lambda \prod_{cyc} \frac{w_a}{r_a} \ge \lambda + \frac{3}{2} \;\forall \,\Delta\, ABC \text{ and } \forall\; \lambda \le \frac{1}{2},$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3783. In ΔABC the following relationship holds:

$$\frac{13}{4} - \frac{r}{2R} \le \sum \frac{r_b r_c}{w_a^2} \le \frac{3R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$w_a^2 = \frac{4bcs(s-a)}{(b+c)^2}$$

$$\frac{r_b r_c}{w_a^2} = \frac{\frac{F^2}{(s-b)(s-c)}}{\frac{4bcs(s-a)}{(b+c)^2}} = \frac{F^2}{s(s-a)(s-b)(s-c)} \cdot \frac{(b+c)^2}{4bc} = \frac{(b+c)^2}{4bc}$$

$$\frac{r_b r_c}{w_a^2} = \frac{(b+c)^2}{4bc} = \frac{a(b+c)^2}{4abc} = \frac{a(a^2+b^2+c^2) + 2abc - a^3}{4abc} =$$

$$= \frac{1}{4}\left(\frac{a(a^2+b^2+c^2) - a^3}{abc} + 2\right)$$

$$\sum \frac{r_b r_c}{w_a^2} = \frac{1}{4}\left(\frac{\left(\sum a . \sum a^2 - \sum a^3\right)}{abc} + 6\right) =$$

$$=\frac{1}{4}\left(\frac{2s.2(s^2-r^2-4Rr)-2s(s^2-3r^2-6Rr)}{4Rrs}+6\right)=$$

$$=\frac{1}{4}\left(\frac{s^2+r^2+10Rr}{2Rr}\right)\overset{Gerretsen}{\geq}\frac{1}{4}\left(\frac{16Rr-5r^2+r^2+10Rr}{2Rr}\right)=$$

$$=\frac{(26Rr-4r^2)}{8Rr}=\frac{13}{4}-\frac{r}{2R}$$

$$\sum\frac{r_br_c}{w_a^2}=\frac{1}{4}\sum\frac{(b+c)^2}{bc}=\frac{1}{4}\sum\left(\frac{b}{c}+\frac{c}{b}+2\right)\overset{Bandila\,\&\,Euler}{\leq}\frac{1}{4}\sum\left(\frac{R}{r}+\frac{R}{r}\right)=\frac{3R}{2r}$$

Equality holds for an equilateral triangle.

3784. In ΔABC the following relationship holds:

$$\sum_{cyc}h_a sin^2A\leq\sum_{cyc}r_a sin^2A$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$In\ \Delta ABC\ \ wlog\ \ a\leq b\leq c\ ,\qquad h_a\geq h_b\geq h_c$$
$$and\ \ r_a\leq r_b\leq r_c$$

$$LHS=\sum_{cyc}h_a sin^2A\ \overset{Chebyshev}{\overbrace{\leq}}\ \frac{1}{3}\sum_{cyc}h_a\cdot\sum_{cyc}sin^2A\quad(*)$$

$$RHS=\sum_{cyc}r_a sin^2A\ \overset{Chebyshev}{\overbrace{\geq}}\ \frac{1}{3}\sum_{cyc}r_a\cdot\sum_{cyc}sin^2A\quad(**)$$

$$Let's\ prove\ that:\ \sum_{cyc}h_a\leq\sum_{cyc}r_a$$

$$For\ \ this\ :$$

$$\sum_{cyc}h_a=\frac{S^2+r^2+4Rr}{2R}\overset{Gerretsen}{\overbrace{\leq}}\frac{4R^2+4Rr+3r^2+r^2+4Rr}{2R}=$$

$$=\frac{4R^2+8Rr+4r^2}{2R}=\frac{2(r+R)^2}{R}\ ;\ \sum_{cyc}r_a=4R+r$$

$$\frac{2(r+R)^2}{R}\overset{?}{\overbrace{\leq}}4R+r\to\ 2R^2+4Rr+2r^2\overset{?}{\overbrace{\leq}}4R+r\ \to$$

$$2R^2-3Rr-2r^2\overset{?}{\overbrace{\geq}}0\ \ \to\ (R-2r)(2R+r)\geq0\to\ \ R\geq2r\ (EULER)$$

$$Equality\ \ holds\ for\ :\ \ a=b=c$$

3785. In any Δ ABC the following relationship holds :

$$\frac{1}{2R^2r} \le \sum_{cyc} \frac{1}{bc(h_a - r)} \le \frac{R}{16r^4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{bc(h_a - r)} = \sum_{cyc} \frac{1}{bc\left(\frac{2rs}{a} - r\right)} = \sum_{cyc} \frac{a^2}{rabc(b+c)}$$

$$= \frac{1}{4Rr^2s} \cdot \frac{1}{2s(s^2+2Rr+r^2)} \cdot \sum_{cyc}\left(a^2\left(a^2 + \sum_{cyc} ab\right)\right)$$

$$= \frac{1}{8Rr^2s^2(s^2+2Rr+r^2)} \cdot \left(2\sum_{cyc} a^2b^2 - 16r^2s^2 + \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2\right)\right)$$

$$= \frac{(s^2+4Rr+r^2)^2 - 16Rrs^2 - 8r^2s^2 + s^4 - (4Rr+r^2)^2}{4Rr^2s^2(s^2+2Rr+r^2)} \overset{?}{\le} \frac{R}{16r^4}$$

$$\Leftrightarrow (R^2 - 8r^2)s^2 + r(2R^3 + R^2r + 32Rr^2 + 24r^3) \underbrace{\overset{?}{\ge}}_{(*)} 0 \text{ and it's trivially true when :}$$

$$R^2 - 8r^2 \ge 0 \text{ and when : } R^2 - 8r^2 < 0, \text{LHS of } (*) \overset{\text{Gerretsen}}{\ge}$$

$$(R^2 - 8r^2)(4R^2 + 4Rr + 3r^2) + r(2R^3 + R^2r + 32Rr^2 + 24r^3) \overset{?}{\ge} 0$$

$$\Leftrightarrow 2R^2(R-2r)(2R+7r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc} \frac{1}{bc(h_a - r)} \le \frac{R}{16r^4} \text{ and again, } \sum_{cyc} \frac{1}{bc(h_a - r)} \overset{?}{\ge} \frac{1}{2R^2r} \Leftrightarrow$$

$$\frac{(s^2+4Rr+r^2)^2 - 16Rrs^2 - 8r^2s^2 + s^4 - (4Rr+r^2)^2}{4Rr^2s^2(s^2+2Rr+r^2)} \overset{?}{\ge} \frac{1}{2R^2r}$$

$$\Leftrightarrow (R-r)s^2 \underbrace{\overset{?}{\ge}}_{(**)} r(4R^2 + 5Rr + r^2)$$

$$\text{Indeed, } (R-r)s^2 \overset{\text{Gerretsen}}{\ge} (R-r)(16Rr - 5r^2) \overset{?}{\ge} r(4R^2 + 5Rr + r^2)$$

$$\Leftrightarrow 2r(R-2r)(6R-r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r \Rightarrow (**) \text{ is true}$$

$$\therefore \frac{1}{2R^2r} \le \sum_{cyc} \frac{1}{bc(h_a - r)} \le \frac{R}{16r^4} \; \forall \Delta \text{ ABC}, '' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

3786. In any Δ ABC the following relationship ho$l$ds :

$$\sum_{cyc}\frac{1}{bc(h_a-r)}\ge\sum_{cyc}\frac{1}{bc(r_a-r)}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{bc(h_a-r)}=\sum_{cyc}\frac{1}{bc\left(\frac{2rs}{a}-r\right)}=\sum_{cyc}\frac{a^2}{rabc(b+c)}=$$

$$=\frac{1}{4Rr^2s}\cdot\frac{1}{2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left(a^2\left(a^2+\sum_{cyc}ab\right)\right)$$

$$=\frac{1}{8Rr^2s^2(s^2+2Rr+r^2)}\cdot\left(2\sum_{cyc}a^2b^2-16r^2s^2+\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)\right)$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2-8r^2s^2+s^4-(4Rr+r^2)^2}{4Rr^2s^2(s^2+2Rr+r^2)}\overset{?}{\ge}\sum_{cyc}\frac{1}{bc(r_a-r)}$$

$$=\sum_{cyc}\frac{1}{bc\left(\frac{rs}{s-a}-\frac{rs}{s}\right)}=\sum_{cyc}\frac{s-a}{r.4Rrs}=\frac{1}{4Rr^2}\Leftrightarrow s^2(s^2-10Rr-7r^2)\overset{?}{\ge}0$$

$$\Leftrightarrow s^2\left(s^2-16Rr+5r^2+6r(R-2r)\right)\overset{?}{\ge}0\to\text{true}\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\ge}0$$

$$\text{and } R\overset{\text{Euler}}{\ge}2r\therefore\sum_{cyc}\frac{1}{bc(h_a-r)}\ge\sum_{cyc}\frac{1}{bc(r_a-r)}\ \forall\Delta\,\text{ABC},$$

$$''=''\ \text{iff }\Delta\,\text{ABC is equilateral (QED)}$$

3787. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sec(A)}{1+\tan(A)}\ge\frac{6}{1+\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{\sec(A)}{1+\tan(A)}=\sum_{cyc}\frac{\frac{1}{\cos(A)}}{1+\tan(A)}=\sum_{cyc}\frac{1}{\sin(A)+\cos(A)}\to$$

$$A\in\left(0;\frac{\pi}{2}\right)\quad=\frac{1}{\sqrt{2}}\sum_{cyc}\frac{1}{\sin\left(\frac{\pi}{4}+A\right)}$$

$$Let\ \ f(x)=\frac{1}{\sin(x)},\qquad x\in\left(\frac{\pi}{4};\frac{3\pi}{4}\right),\qquad f'(x)=-\frac{\cos(x)}{sin^2(x)},$$

$$f''(x)=\frac{1+cos^2(x)}{sin^3(x)}>0\ \ So,\qquad f(x)\ \ is\ \ convexe\ function$$

$$By\ \ Jensen's\ \ inequality:$$

$$\sum_{cyc}\frac{1}{\sin(A)+\cos(A)}\ge 3.\frac{1}{\sin\left(\frac{A+B+C}{3}\right)+\cos\left(\frac{A+B+C}{3}\right)}=\frac{3}{\frac{\sqrt{3}}{2}+\frac{1}{2}}=\frac{6}{1+\sqrt{3}}$$

$$Equality\ \ holds\ \ for\ \ an\ \ equilateral\ \ triangle.$$

3788. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\csc\left(\frac{A}{2}\right)\left(1+\tan\left(\frac{A}{2}\right)\right)\ge 6+2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\csc\left(\frac{A}{2}\right)\left(1+\tan\left(\frac{A}{2}\right)\right)=\sum_{cyc}\left(\frac{1}{\sin\left(\frac{A}{2}\right)}+\frac{1}{\cos\left(\frac{A}{2}\right)}\right)$$

$$Let\ \ f(x)=\frac{1}{\sin(x)}+\frac{1}{\cos(x)},\quad \frac{A}{2}=x,\ \ x\in\left(0;\frac{\pi}{2}\right)$$

$$f'(x)=\left(\frac{1}{\sin(x)}+\frac{1}{\cos(x)}\right)'=-\frac{\cos(x)}{sin^2(x)}+\frac{\sin(x)}{cos^2(x)},$$

$$f''(x)=\frac{1+cos^2(x)}{sin^3(x)}+\frac{1+sin^2(x)}{cos^3(x)}>0$$

$$f(x)\ \ is\ \ convexe\ function.\ By\ \ Jensen's\ \ inequality:$$

$$\sum_{cyc}\left(\frac{1}{\sin\left(\frac{A}{2}\right)}+\frac{1}{\cos\left(\frac{A}{2}\right)}\right)\ge 3.\frac{1}{\sin\left(\frac{A+B+C}{6}\right)+\cos\left(\frac{A+B+C}{6}\right)}=$$

$$3\left(\frac{1}{\sin\left(\frac{\pi}{6}\right)}+\frac{1}{\cos\left(\frac{\pi}{6}\right)}\right)=3\left(2+\frac{2}{\sqrt{3}}\right)=6+2\sqrt{3}$$

$$Equality\ \ holds\ \ for\ \ an\ equilateral\ \ triangle.$$

3789. In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{w_a+w_b}{w_c}}+\frac{1013R^3}{4r^3}\ge 2026+\sum\sqrt{\frac{m_a+m_b}{m_c}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\sum\sqrt{\frac{w_a+w_b}{w_c}}\overset{AM-GM}{\ge}\sum\frac{\sqrt{2}\sqrt[4]{w_aw_b}}{\sqrt{w_c}}\overset{AM-GM}{\ge}3\sqrt{2}$$

$$\frac{(m_a+m_b)}{m_c}=\frac{m_a}{m_c}+\frac{m_b}{m_c}\overset{Panaitopol\ \&\ m_c\ge h_c}{\le}\frac{\frac{Rh_a}{2r}}{h_c}+\frac{\frac{Rh_b}{2r}}{h_c}=\frac{R}{2r}\left(\frac{h_a}{h_c}+\frac{h_b}{h_c}\right)=\frac{R}{2r}\left(\frac{c}{a}+\frac{c}{b}\right)$$

$$\sum\sqrt{\frac{m_a+m_b}{m_c}}\le\sqrt{\frac{R}{2r}}\sum\sqrt{\left(\frac{c}{a}+\frac{c}{b}\right)}\overset{CBS}{\le}\sqrt{\frac{R}{2r}}\sqrt{3\sum\left(\frac{c}{a}+\frac{c}{b}\right)}=$$

$$=\sqrt{\frac{R}{2r}}\sqrt{3\sum\left(\frac{c}{a}+\frac{a}{c}\right)}\overset{Bandila}{\le}\sqrt{\frac{R}{2r}}\sqrt{\frac{9R}{r}}=\frac{3\sqrt{2}R}{r}$$

We need to show:

$$\sum\sqrt{\frac{w_a+w_b}{w_c}}+\frac{1013R^3}{4r^3}\ge 2026+\sum\sqrt{\frac{m_a+m_b}{m_c}}$$

$$3\sqrt{2}+\frac{1013R^3}{4r^3}\ge 2026+\frac{3\sqrt{2}R}{r}$$

$$1013x^3-6\sqrt{2}x+12\sqrt{2}-8104\overset{\frac{R}{r}=x\ge 2\ Euler}{\ge}0$$

$$(x-2)\left(1013(x^2+2x+4)-6\sqrt{2}\right)\ge 0\ true\ as\ x\ge 2$$

Equality holds for an equilateral triangle.

3790.

In ΔABC: I_a,I_b,I_c excenters of ΔABC
(H_A,R_a) (H_B,R_b) (H_C,R_c) orthococenters and circumradius of $(\Delta I_aBC),(\Delta I_bAC)$ respectively (ΔI_cAB). Prove that:

$$\frac{I_aH_A}{R_a}+\frac{I_bH_B}{R_b}+\frac{I_cH_C}{R_c}\le 3$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Amin Hajiyev-Azerbaijan

Lemma 1: *In ΔABC with orthocenter H and circumradius R*

$$AH = 2R|\cos(A)|$$

Lemma 2: $\angle BI_aC = 90° - \frac{A}{2}, \angle I_aBC = 90° - \frac{B}{2}, I_aCB = 90° - \frac{C}{2}$

Lemma 3: $f(x) = \sin\left(\frac{x}{2}\right) \quad x \in (0;\pi) \to \sin\left(\frac{x}{2}\right) > 0$

$$\frac{d^2}{dx^2}f(x) = -\frac{1}{4}\sin\left(\frac{x}{2}\right) \to \quad f''(x) < 0 \text{ concave function on } (0;\pi)$$

$$\frac{\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right)}{3} \overset{JENSEN}{\leq} \sin\left(\frac{A+B+C}{6}\right) = \sin\left(\frac{\pi}{6}\right)$$

$$\sin\left(\frac{A}{2}\right) + \sin\left(\frac{B}{2}\right) + \sin\left(\frac{C}{2}\right) \leq \frac{3}{2}$$

$$I_aH_A = 2R_a|\cos(\angle BI_aC)| \overset{Lemma\ 2}{=} 2R_a\cos\left(90° - \frac{A}{2}\right) = 2R_a\sin\left(\frac{A}{2}\right)$$

$$\frac{I_aH_A}{R_a} = 2\sin\left(\frac{A}{2}\right) \to \sum_{cyc}\frac{I_aH_A}{R_a} = 2\underbrace{\sum_{cyc}\sin\left(\frac{A}{2}\right)}\overset{Lemma\ 3}{\leq} 2\cdot\frac{3}{2} = 3$$

$$\frac{I_aH_A}{R_a} + \frac{I_bH_B}{R_b} + \frac{I_cH_C}{R_c} \leq 3$$

3791. ***In ΔABC holds :***

$$\frac{1}{bc(r_a - r)} + \frac{1}{ac(r_b - r)} + \frac{1}{ab(r_c - r)} \leq \left(\frac{1}{2r}\right)^3$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Amin Hajiyev-Azerbaijan

Lemma 1. $S = pr = (p-a)r_a = (p-b)r_b = (p-c)r_c, \qquad abc = 4RF$

Lemma 2. $r_a - r = \frac{ar}{p-a}, \qquad r_a = \frac{S}{p-a}, \qquad r = \frac{F}{p}, r_a - r = \frac{aF}{p(p-a)}$

Lemma 3. $R \overset{Euler}{\geq} 2r \xrightarrow{Lemma\ 2} \begin{cases} \frac{1}{bc(r_a - r)} = \frac{p-a}{abcr} \\ \frac{1}{ac(r_b - r)} = \frac{p-b}{abcr} \\ \frac{1}{ab(r_c - r)} = \frac{p-c}{abcr} \end{cases}$

$$LHS = \sum_{cyc}\frac{1}{bc(r_a - r)} = \frac{3p - (a+b+c)}{abcr} = \frac{3p - 2p}{abcr} = \frac{p}{abcr}$$

$$LHS \overset{Lemma\ 1;2}{=} \frac{\frac{S}{r}}{4RSr} = \frac{1}{4Rr^2} \quad R \overset{Euler}{\geq} 2r \quad \frac{1}{R} \leq \frac{1}{2r} \to \frac{1}{4Rr^2} \leq \frac{1}{8r^3}$$

$$\sum_{cyc}\frac{1}{bc(r_a-r)}\le\left(\frac{1}{2r}\right)^3$$

3792. *If I_a, I_b, I_c – excenters then in ΔABC the following relationship holds:*

$$\frac{AI_a}{\sin\frac{A}{2}}+\frac{BI_b}{\sin\frac{B}{2}}+\frac{CI_c}{\sin\frac{C}{2}}\ge 36r$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Ertan Yildirim-Turkiye

Lemma: $AI_a=\dfrac{s}{\cos\frac{A}{2}}$

Proof: *We know that the points A, I and I_a are collinear. Therefore,* $\angle I_aAC=\dfrac{A}{2}$

Let F be the point where the excircle opposite A in tangent to AC. Then

$$\cos\frac{A}{2}=\frac{AF}{AI_a}.\ Since:\ AF=b+s-b=s, it\ follows\ that\ AI_a=\frac{s}{\cos\frac{A}{2}}$$

$$LHS=\sum\frac{AI_a}{\sin\left(\frac{A}{2}\right)}=\sum\left(\frac{2s}{sinA}\right)=2s\sum\frac{2R}{a}=4Rs\sum\left(\frac{1}{a}\right)\ge 4Rs.\left(\frac{\sqrt{3}}{R}\right)=$$

$$\ge 4\sqrt{3}s\ge 4\sqrt{3}.3\sqrt{3}r=36r.$$

Equality holds for $a=b=c$.

3793. *In ΔABC, I – incenter. O_A, O_B, O_C circumcenters of $\Delta BIC, \Delta AIC, \Delta AIB$. Prove that:*

$$[AO_BC]+[AO_CB]+[BO_AC]\ge 3\sqrt{3}r^2$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by proposer

$$Let\ \angle BO_AC = 180 - A, \Delta BO_AC \Rightarrow 2R_A = \frac{a}{\sin\left(90+\frac{A}{2}\right)} \Rightarrow R_A = \frac{a}{2\cos\left(\frac{A}{2}\right)}.$$

$$[BO_AC] = \frac{1}{2}.R_A^2.\sin(180-A) = \frac{1}{2}.\frac{a^2}{4\cos\left(\frac{A}{2}\right)^2}.sinA = \frac{a^2}{4}.\tan\left(\frac{A}{2}\right)$$

$$LHS = \sum\frac{a^2}{4}.\tan\left(\frac{A}{2}\right) = \frac{1}{4}.\sum a^2.\tan\left(\frac{A}{2}\right) = \frac{1}{4}\sum a^2.\left(\frac{r_a}{s}\right) = \frac{1}{4s}.\sum\left(\frac{a^2}{\frac{1}{r_a}}\right) \geq$$

$$\geq \frac{1}{4s}.\frac{4s^2}{\frac{1}{r}} = sr \geq 3\sqrt{3}r^2$$

$$Equality\ holds\ for\ a = b = c.$$

3794. In any acute Δ ABC the following relationship holds :

$$\frac{AH}{r} + \frac{n_a}{n_a - \sqrt{4r^2 + (b-c)^2}} \geq \frac{n_b}{h_b} + \frac{m_a}{h_c} + \frac{m_c}{h_a} + 2$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$n_a^2 \overset{\text{Bogdan Fustei}}{=} s^2 - 2r_ah_a \therefore a^2n_a^2 \overset{?}{\leq} 4(R-r)^2s^2$$

$$\Leftrightarrow a^2(s^2 - 2h_ar_a) \overset{?}{\leq} 4(R-r)^2s^2$$

$$\Leftrightarrow (4R^2\sin^2 A)s^2 - 4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right) \overset{?}{\leq} 4(R^2 - 2Rr + r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2 A) - 2Rr\left(1 - 2\sin^2\frac{A}{2}\right) + r^2 \overset{?}{\geq} 0 \Leftrightarrow R^2\cos^2 A - 2Rr\cos A + r^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R\cos A - r)^2 \overset{?}{\geq} 0 \to \text{true} \therefore n_a \leq \frac{2(R-r)s}{a} = \frac{R-r}{r}.\frac{2rs}{a} \Rightarrow \frac{n_a}{h_a} \leq \frac{R}{r} - 1$$

$$and\ analogs \therefore \frac{n_b}{h_b} + \frac{m_a}{h_c} + \frac{m_c}{h_a} + 2 \leq \frac{R}{r} + 1 + \frac{m_a}{h_c} + \frac{m_c}{h_a} \leq \frac{2R}{r} + 1$$

$$\left(\begin{gathered}\because \frac{R}{r} \geq \frac{m_b}{h_c} + \frac{m_c}{h_b}\ and\ analogs \to \text{reference : } article\ titled \\ \text{"New Triangle Inequalities With Brocard's Angle" by Bogdan Fustei,} \\ \text{Mohamed Amine Ben Ajiba; Lemma 12, 6 – 7, published at : www.ssmrmh.ro}\end{gathered}\right)$$

$$\overset{?}{=} \frac{AH}{r} + \frac{n_a}{n_a - \sqrt{4r^2 + (b-c)^2}} \overset{\text{Bogdan Fustei}}{=} \frac{2R\cos A}{r} + \frac{n_a}{n_a - \frac{an_a}{s}}$$

$$(\because \Delta\ ABC\ \text{is acute} \Rightarrow |2R\cos A| = 2R\cos A) = \frac{2R\cos A}{r} + \frac{s-a+a}{s-a}$$

$$=\frac{2R\cos A}{r}+1+\frac{a}{s-a}\Leftrightarrow\frac{2R}{r}\stackrel{?}{=}\frac{2R\cos A}{r}+\frac{a}{s-a}$$

$$\Leftrightarrow\frac{2R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\left(2\sin^2\frac{A}{2}\right)\stackrel{?}{=}\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}\to\text{true}$$

$$\therefore\frac{AH}{r}+\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\geq\frac{n_b}{h_b}+\frac{m_a}{h_c}+\frac{m_c}{h_a}+2\ \forall\,\Delta\,ABC,''=''\text{ iff } c=a\ (QED)$$

3795. In any Δ ABC the following relationship holds :

$$\sqrt{3}.\left(\frac{2r}{R}\right)\leq\sum_{cyc}\frac{\sin A+\sin B}{(\sin B+\sin C)(\sin C+\sin A)}\leq\sqrt{3}\left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\sin A+\sin B}{(\sin B+\sin C)(\sin C+\sin A)}=2R.\sum_{cyc}\frac{(a+b)^2}{(a+b)(b+c)(c+a)}$$

$$=\frac{4R}{2s(s^2+2Rr+r^2)}.\left(\sum_{cyc}a^2+\sum_{cyc}ab\right)$$

$$\therefore\sum_{cyc}\frac{\sin A+\sin B}{(\sin B+\sin C)(\sin C+\sin A)}=\frac{2R(3s^2-4Rr-r^2)}{s(s^2+2Rr+r^2)}\to(1)\text{ and}$$

$$\because s\overset{\text{Mitrinovic}}{\geq}3\sqrt{3}r\ \therefore\text{ in order to prove :}$$

$$\sum_{cyc}\frac{\sin A+\sin B}{(\sin B+\sin C)(\sin C+\sin A)}\stackrel{?}{\leq}\sqrt{3}.\left(\frac{R}{2r}\right)^2\text{, it suffices to prove :}$$

$$9R(s^2+2Rr+r^2)\stackrel{?}{\geq}8r(3s^2-4Rr-r^2)$$

$$\Leftrightarrow(9R-24r)s^2+r(18R^2+41Rr+8r^2)\underbrace{\stackrel{?}{\geq}}_{(*)}0\text{ and it's trivially true when :}$$

$$9R-24r\geq0\text{ and when : }9R-24r<0,\ LHS\ of\ (*)\overset{\text{Gerretsen}}{\geq}$$

$$(9R-24r)(4R^2+4Rr+3r^2)+r(18R^2+41Rr+8r^2)\stackrel{?}{\geq}0$$

$$\Leftrightarrow18t^3-21t^2-14t-32\stackrel{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(18t^2+15t+16)\stackrel{?}{\geq}0\to\text{true}$$

$$\because t\overset{\text{Euler}}{\geq}2\Rightarrow(*)\text{ is true }\forall\,\Delta\,ABC\ \therefore\sum_{cyc}\frac{\sin A+\sin B}{(\sin B+\sin C)(\sin C+\sin A)}\leq\sqrt{3}.\left(\frac{R}{2r}\right)^2$$

$$\text{Again, since } s \overset{\text{Mitrinovic}}{\leq} \frac{3\sqrt{3}R}{2} \therefore (1) \Rightarrow \text{in order to prove :}$$

$$\sum_{\text{cyc}} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} \overset{?}{\geq} \sqrt{3}.\left(\frac{2r}{R}\right), \text{it suffices to prove :}$$

$$2R(3s^2 - 4Rr - r^2) \overset{?}{\geq} 9r(s^2 + 2Rr + r^2)$$

$$\Leftrightarrow (6R - 9r)s^2 - r(8R^2 + 20Rr + 9r^2) \underbrace{\overset{?}{\geq}}_{(**)} 0 \text{ and once again, LHS of } (**) \overset{\text{Gerretsen}}{\geq}$$

$$(6R - 9r)(16Rr - 5r^2) - r(8R^2 + 20Rr + 9r^2) = 2r(R - 2r)(44R - 9r) \geq 0$$

$$\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \text{ is true } \therefore \sum_{\text{cyc}} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} \geq \sqrt{3}.\left(\frac{2r}{R}\right)$$

$$\text{and so, } \sqrt{3}.\left(\frac{2r}{R}\right) \leq \sum_{\text{cyc}} \frac{\sin A + \sin B}{(\sin B + \sin C)(\sin C + \sin A)} \leq \sqrt{3}.\left(\frac{R}{2r}\right)^2 \ \forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3796. In any Δ ABC the following relationship holds :

$$n_a + n_b + n_c \leq p_a + p_b + p_c + \frac{4}{5}(\max\{a, b, c\} - \min\{a, b, c\})$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$n_a - p_a \leq \frac{s|b - c|}{2s + a} \text{ and analogs}$$

(Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba – 52; published at www.ssmrmh.ro)

$$\therefore \sum_{\text{cyc}} n_a - \sum_{\text{cyc}} p_a \leq \sum_{\text{cyc}} \frac{s|b - c|}{2s + a} \overset{\text{CBS}}{\leq} s.\sqrt{\sum_{\text{cyc}} \frac{(b - c)^2}{2s + a}}.\sqrt{\sum_{\text{cyc}} \frac{1}{2s + a}}$$

$$= s.\sqrt{\frac{1}{2s(9s^2 + 6Rr + r^2)} \sum_{\text{cyc}} \left((8s^2 - 2sa + bc)\left(\sum_{\text{cyc}} a^2 - a^2 - 2bc\right)\right)}.\sqrt{\frac{21s^2 + 4Rr + r^2}{2s(9s^2 + 6Rr + r^2)}}$$

$$= s.\sqrt{\frac{\begin{matrix} 2(s^2 - 4Rr - r^2)(24s^2 - 4s^2 + s^2 + 4Rr + r^2) - \\ 16s^2(s^2 - 4Rr - r^2) + 4s^2(s^2 - 6Rr - 3r^2) - \\ 8Rrs^2 - 16s^2(s^2 + 4Rr + r^2) + 48Rrs^2 - \\ 2((s^2 + 4Rr + r^2)^2 - 16Rrs^2) \end{matrix}}{2s(9s^2 + 6Rr + r^2)}}.\sqrt{\frac{21s^2 + 4Rr + r^2}{2s(9s^2 + 6Rr + r^2)}}$$

$$\therefore \sum_{cyc} n_a - \sum_{cyc} p_a \overset{①}{\leq} s.\sqrt{\frac{(3s^4 - (32Rr + 14r^2)s^2 - r^2(4R+r)^2)(21s^2 + 4Rr + r^2)}{4s^2(9s^2 + 6Rr + r^2)^2}}$$

$$\text{and again,} \frac{4}{5}(max\{a,b,c\} - min\{a,b,c\}) = \frac{2}{5}\sum_{cyc}|b-c|$$

$$\left(\begin{array}{c}\text{Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba} - 45;\\ \text{published at www.ssmrmh.ro}\end{array}\right)$$

$$= \frac{2}{5}.\sqrt{\sum_{cyc}(b-c)^2 + 2\sum_{cyc}(|c-a||a-b|)} \overset{\text{Triangle Inequality}}{\geq}$$

$$\frac{2}{5}.\sqrt{2(s^2 - 12Rr - 3r^2) + 2\left|\sum_{cyc}(c-a)(a-b)\right|}$$

$$= \frac{2}{5}.\sqrt{2(s^2 - 12Rr - 3r^2) + 2\left|\sum_{cyc} a^2 - \sum_{cyc} ab\right|} = \frac{2}{5}.\sqrt{4(s^2 - 12Rr - 3r^2)}$$

$$\overset{②}{=} \frac{4}{5}(max\{a,b,c\} - min\{a,b,c\}) \therefore ① \text{ and } ② \Rightarrow \text{it suffices to prove :}$$

$$\frac{(3s^4 - (32Rr + 14r^2)s^2 - r^2(4R+r)^2)(21s^2 + 4Rr + r^2)}{4(9s^2 + 6Rr + r^2)^2} \overset{?}{\leq} \frac{16(s^2 - 12Rr - 3r^2)}{25}$$

$$\Leftrightarrow 3609s^6 - (38796Rr + 7125r^2) - r^2(69040R^2 + 27392Rr + 2517r^2)s^2 -$$

$$r^3(26048R^3 + 14928R^2r + 2772Rr^2 + 167r^3) \boxed{\overset{(\bullet)}{\geq}} 0 \text{ and } \because P =$$

$$3609(s^2 - 16Rr + 5r^2)^3 + 4r(33609Rr - 15315r)(s^2 - 16Rr + 5r^2)^2 +$$

$$8r^2\left(26R^2 - Rr - 6r^2 + 32(5707R^2 - 6249Rr + 1326r^2)\right) \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{in order to prove } (\bullet), \text{it suffices to prove : LHS of } (\bullet) \overset{?}{\geq} P$$

$$\Leftrightarrow 26t^3 + 13t^2 + 4t - 12 + 32(3632t^3 - 9359t^2 + 4487t - 602) \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(26t^2 + t + 6 + 32(3632t^2 - 2093t + 301)\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\Rightarrow (\bullet) \text{ is true} \therefore n_a + n_b + n_c \leq p_a + p_b + p_c + \frac{4}{5}(max\{a,b,c\} - min\{a,b,c\})$$

$$\forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3797. In any ΔABC the following relationship holds :

$$\frac{p_a - m_a}{a} + \frac{p_b - m_b}{b} + \frac{p_c - m_c}{c} < \frac{1}{3}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\mathbf{p}_a - \mathbf{m}_a \le \frac{a|b-c|}{2(2s+a)} \text{ and analogs}$$

$$\left(\begin{matrix}\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba} - 56;\\ \text{published at www.ssmrmh.ro}\end{matrix}\right)$$

$$\therefore \sum_{cyc}\frac{p_a - m_a}{a} < \sum_{cyc}\frac{|b-c|}{2(2s+a)} \Rightarrow \left(\sum_{cyc}\frac{p_a - m_a}{a}\right)^2 <$$

$$\frac{1}{4}\left(\sum_{cyc}\frac{(b-c)^2}{(2s+a)^2} + 2\sum_{cyc}\frac{|b-c||c-a|}{(2s+a)(2s+b)}\right) <$$

$$\frac{1}{4}\cdot\left(\begin{matrix}\frac{1}{4s^2(9s^2+6Rr+r^2)^2}\cdot\sum_{cyc}\left((b-c)^2(2s+b)^2(2s+c)^2\right)+\\ \frac{2}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc}\left(ab(2s+c)\right)\end{matrix}\right)$$

$$\therefore \left(\sum_{cyc}\frac{p_a - m_a}{a}\right)^2 \overset{①}{<} \frac{1}{4}\left(\frac{1}{4s^2(9s^2+6Rr+r^2)^2}\cdot\left(\begin{matrix}\sum_{cyc}\left((b-c)^2(2s+b)^2(2s+c)^2\right)+\\ 8s^2(9s^2+6Rr+r^2)(s^2+10Rr+r^2)\end{matrix}\right)\right)$$

$$\text{Now, }\sum_{cyc}\left((b-c)^2(2s+b)^2(2s+c)^2\right) = \sum_{cyc}\left((b-c)^2(8s^2-2sa+bc)^2\right)$$

$$= \sum_{cyc}\left((b-c)^2(64s^4+16bcs^2+b^2c^2-32s^3a-4abcs+4a^2s^2)\right)$$

$$= 128s^4(s^2-12Rr-3r^2)+$$
$$16s^2\left(2(s^4-(4Rr+r^2)^2)-8Rrs^2-2((s^2+4Rr+r^2)^2-16Rrs^2)\right)+$$
$$2(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16Rrs^2)-48R^2r^2s^2-$$
$$2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)-64s^4(s^2-14Rr+r^2)-$$
$$32Rrs^2(s^2-12Rr-3r^2)+8s^2((s^2+4Rr+r^2)^2-16Rrs^2)-64Rrs^4$$

$$\left(\because (b-c)^2 = \sum_{cyc}a^2 - a^2 - 2bc\right)$$

$$= 4(18s^6-(168Rr+125r^2)s^4-r^2(116R^2+84Rr+16r^2)-r^3(4R+r)^3)$$

$$\therefore \text{ via } ①, \left(\sum_{cyc}\frac{p_a - m_a}{a}\right)^2 <$$

$$\frac{4(18s^6-(168Rr+125r^2)s^4-r^2(116R^2+84Rr+16r^2)s^2-r^3(4R+r)^3)+8s^2(9s^2+6Rr+r^2)(s^2+10Rr+r^2)}{16s^2(9s^2+6Rr+r^2)^2} \overset{?}{<} \frac{1}{9}$$

$$\Leftrightarrow (216R+1017r)s^4+r(108R^2+516Rr+130r^2)s^2+9r^2(4R+r)^3 \overset{?}{>} 0 \to \text{true}$$

$$\therefore \frac{p_a-m_a}{a}+\frac{p_b-m_b}{b}+\frac{p_c-m_c}{c}<\frac{1}{3}\ \forall\,\Delta\,ABC\ (QED)$$

3798. In any ΔABC the following relationship holds :

$$\sum_{cyc}\frac{\sec\frac{A}{2}}{1+\tan\frac{A}{2}} \ge \frac{6}{1+\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(x)=\frac{\sec\frac{x}{2}}{1+\tan\frac{x}{2}}\ \forall \in (0,\pi) \text{ and then : } f''(x)=$$

$$\frac{\sec^3\frac{x}{2}+\sec\frac{x}{2}\tan\frac{x}{2}\left(\tan^2\frac{x}{2}+\tan\frac{x}{2}-\sec^2\frac{x}{2}\right)}{4\left(1+\tan\frac{x}{2}\right)^2}-\frac{\sec^3\frac{x}{2}\left(\tan^2\frac{x}{2}+\tan\frac{x}{2}-\sec^2\frac{x}{2}\right)}{2\left(1+\tan\frac{x}{2}\right)^3}$$

$$=\frac{3-\sin x}{4\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^3}\ge 0\ \left(\because \sin x\le 1<3 \text{ and } \sin\frac{x}{2},\cos\frac{x}{2}>0 \text{ as } \frac{x}{2}\in\left(0,\frac{\pi}{2}\right)\right)$$

$$\therefore f(x)=\frac{\sec\frac{x}{2}}{1+\tan\frac{x}{2}} \text{ is convex } \forall \in (0,\pi) \text{ and so, } \sum_{cyc}\frac{\sec\frac{A}{2}}{1+\tan\frac{A}{2}} \overset{\text{Jensen}}{\ge} 3.\frac{\sec\frac{\pi}{6}}{1+\tan\frac{\pi}{6}}$$

$$=\frac{6}{1+\sqrt{3}}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3799. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sec\frac{A}{2}}{1+\tan\frac{A}{2}} \ge \frac{6}{1+\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } f(x) = \frac{\sec\frac{x}{2}}{1+\tan\frac{x}{2}}\ \forall \in (0,\pi) \text{ and then : } f''(x) =$$

$$\frac{\sec^3\frac{x}{2} + \sec\frac{x}{2}\tan\frac{x}{2}\left(\tan^2\frac{x}{2} + \tan\frac{x}{2} - \sec^2\frac{x}{2}\right)}{4\left(1+\tan\frac{x}{2}\right)^2} - \frac{\sec^3\frac{x}{2}\left(\tan^2\frac{x}{2} + \tan\frac{x}{2} - \sec^2\frac{x}{2}\right)}{2\left(1+\tan\frac{x}{2}\right)^3}$$

$$= \frac{3-\sin x}{4\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^3} \geq 0\ \left(\because \sin x \leq 1 < 3 \text{ and } \sin\frac{x}{2}, \cos\frac{x}{2} > 0 \text{ as } \frac{x}{2} \in \left(0,\frac{\pi}{2}\right)\right)$$

$$\therefore f(x) = \frac{\sec\frac{x}{2}}{1+\tan\frac{x}{2}} \text{ is convex } \forall \in (0,\pi) \text{ and so, } \sum_{cyc}\frac{\sec\frac{A}{2}}{1+\tan\frac{A}{2}} \overset{\text{Jensen}}{\geq} 3.\frac{\sec\frac{\pi}{6}}{1+\tan\frac{\pi}{6}}$$

$$= \frac{6}{1+\sqrt{3}}\ \forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3800. In ΔABC the following relationship holds:

$$\frac{r_a^{n-1}(r_c^{n+1}+r_a^{n+1})}{r_b^n r_c(r_a^n r_b^n + r_c^{2n})} + \frac{r_b^{n-1}(r_a^{n+1}+r_b^{n+1})}{r_c^n r_a(r_c^n r_b^n + r_a^{2n})} + \frac{r_c^{n-1}(r_c^{n+1}+r_b^{n+1})}{r_a^n r_b(r_a^n r_c^n + r_b^{2n})} \geq \frac{2}{R}\cdot\left(\frac{2}{3R}\right)^n, n \in \mathbb{N}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{r_a^{n-1}(r_c^{n+1}+r_a^{n+1})}{r_b^n r_c(r_a^n r_b^n + r_c^{2n})} = \sum_{cyc}\frac{\left(\frac{r_a}{r_b}\right)^n}{\left(\left(\frac{r_b}{r_c}\right)^n + \left(\frac{r_c}{r_a}\right)^n\right)}\cdot\frac{1}{(r_a r_c)^{n+1}}(r_c^{n+1}+r_a^{n+1}) =$$

$$= \sum_{cyc}\frac{\left(\frac{r_a}{r_b}\right)^n}{\left(\frac{r_b}{r_c}\right)^n + \left(\frac{r_c}{r_a}\right)^n}\left(\frac{1}{r_a^{n+1}} + \frac{1}{r_c^{n+1}}\right)$$

The left side of the inequality satisfies the conditions of Walter Janous' theorem:

x, y, z, a', b', c' – positive real numbers:

$$\frac{x}{y+z}(b'+c') + \frac{y}{x+z}(c'+a') + \frac{z}{x+y}(a'+b') \geq \sqrt{3\sum_{cyc}a'b'}$$

$$Here:\ x = \left(\frac{r_c}{r_a}\right)^n, y = \left(\frac{r_a}{r_b}\right)^n, z = \left(\frac{r_b}{r_c}\right)^n, a' = \frac{1}{r_a^{n+1}}, b' = \frac{1}{r_b^{n+1}}, c' = \frac{1}{r_c^{n+1}}$$

$$LHS=\sqrt{3}\left(\left(\frac{1}{r_ar_b}\right)^{n+1}+\left(\frac{1}{r_cr_b}\right)^{n+1}+\left(\frac{1}{r_ar_c}\right)^{n+1}\right)^{\frac{1}{2}}\overset{AM-GM}{\geq}$$

$$\sqrt{3}.\sqrt{3}\left(\left(\frac{1}{r_ar_br_c}\right)^{2n+2}\right)^{\frac{1}{6}}=3\left(\frac{1}{S^2r}\right)^{\frac{n+1}{3}}\overset{Euler,Mitrinovic}{\geq}$$

$$3\left(\frac{8}{27R^3}\right)^{\frac{n+1}{3}}=\left(\frac{2}{3R}\right)^{n+1}=\frac{2}{R}.\left(\frac{2}{3R}\right)^{n}$$

Equality holds for an equilateral triangle.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru